Unavoidable cycle-contraction minors of large 2-connected graphs

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Clemson, March 2025

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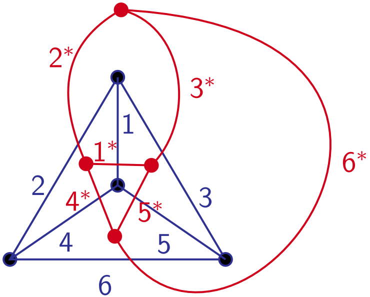
- H is an induced subgraph of G if it can be obtained by a sequence of vertex deletions.
- H is a minor of G if it can be obtained by a sequence of vertex and edge deletions, and edge contractions.

Duality



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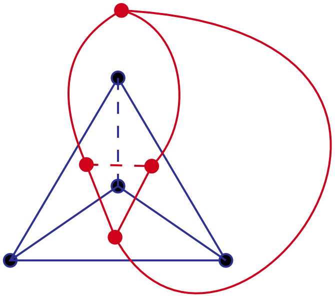

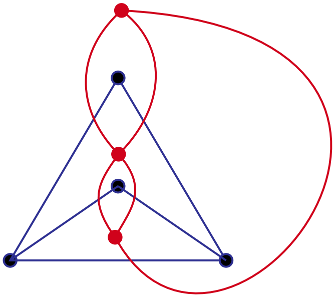

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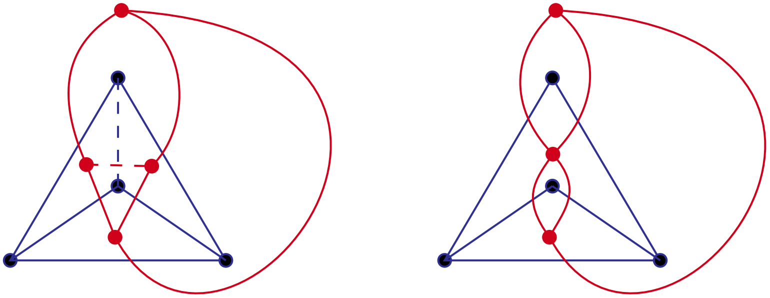

Proposition

Let G be a plane graph, $G \setminus e = (G^*/e^*)^*$.

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Does vertex deletion have a dual operation?

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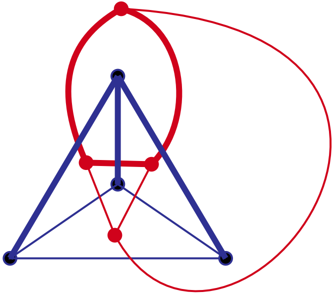
- $E_v^* := \{e^* | e \in E_v\}$ is a cycle in G^*.



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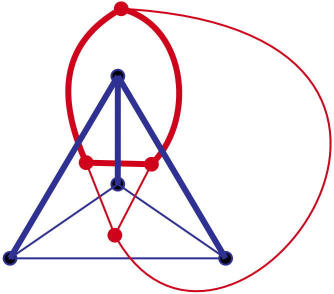

- $G - v = (G^*/E_v^*)^*$.

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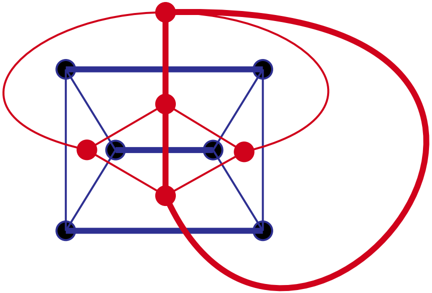
However, not every cycle corresponds to a vertex bond in the dual graph.



Duality

Lemma

Let G be a loopless 2-connected plane graph. A graph H is a 2-connected induced subgraph of G if and only if H^* is a 2-connected cc-minor of G^*.

Ramsey properties and connectivity

Theorem (Ramsey 1930)

Every giant simple graph contains a large induced K_r or $\overline{K_r}$.

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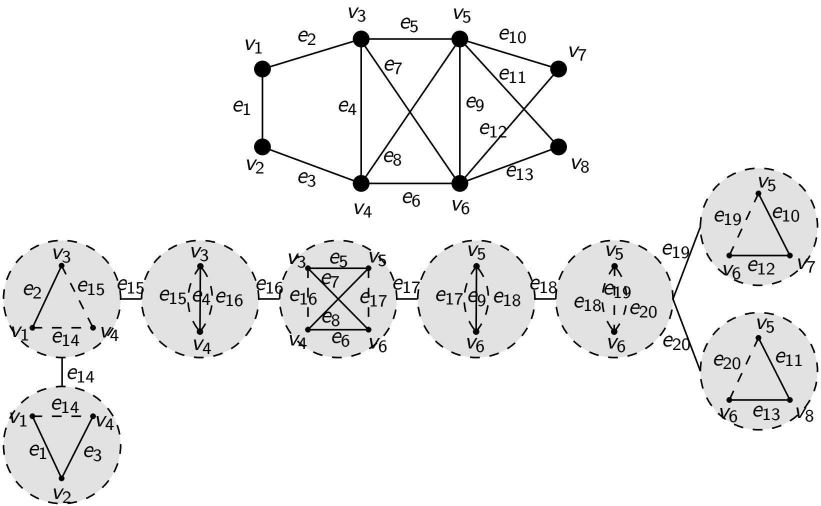
Every giant simple **connected** graph contains a large induced K_r, $K_{1,r}$ or P_r.

Theorem (Allred, Ding, Oporowski 2023)

Every giant simple 2-**connected** graph contains a large induced ...

Tree decomposition of 2-connected graph



Tree decomposition of 2-connected graph

Theorem (Tutte 1966)

Every 2-connected graph G has a tree decomposition in which every vertex label is a simple 3-connected graph, a copy of K_3, or a copy of B_3. Moreover, each vertex label is isomorphic to a minor of G.

r-template

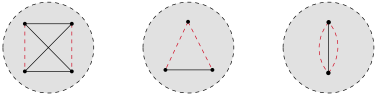
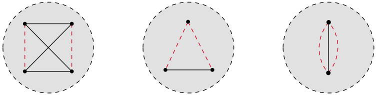


Figure: Building blocks of templates

r-template



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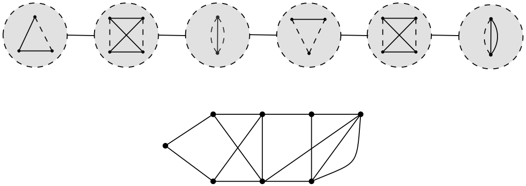
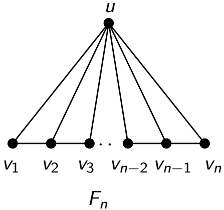


Figure: A sample 6-template

r-template



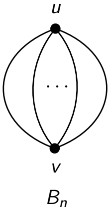
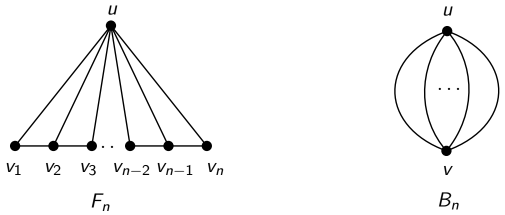


Figure: A fan graph F_n and a bond graph B_n.

Examples

(i) A **fan graph** F_n is a $(2n-3)$-template in which the parts alternate between K_3 and B_3, beginning and ending with K_3.

r-template



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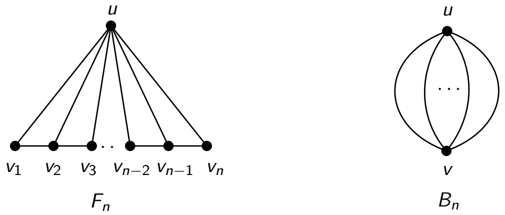

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(iii) For $n \geq 3$, a **cycle** C_n is an $(n-2)$-template for which every part is a K_3.

Ramsey properties and connectivity

Theorem (Allred, Ding, Oporowski 2023)

Every giant simple 2-**connected** graph contains a large induced K_r or an induced subdivision of an r-template.

Main result (a dual theorem)

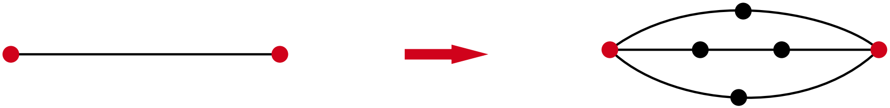

Figure: parallel-path extension

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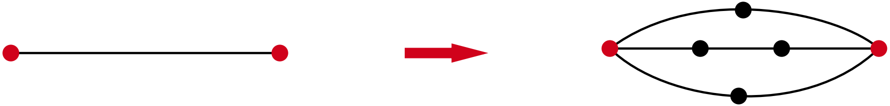

Figure: parallel-path extension

Theorem (G., Oxley 2025+)

Every giant 2-connected graph contains a **parallel-path extension** of a large r-template as a cc-minor.

Proving the main result

Case I

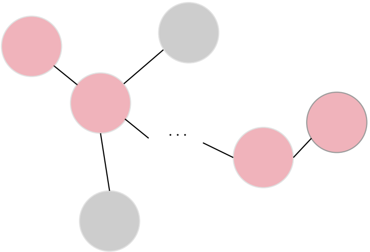

Figure: The tree decomposition has a long path

Case II

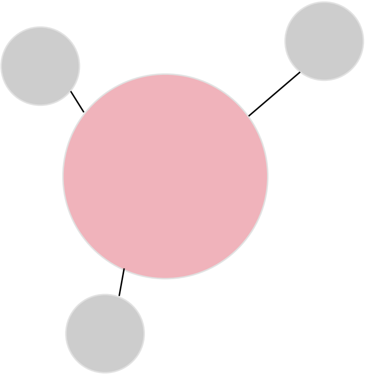

Figure: The tree decomposition has a giant vertex

Proving the main result

Lemma

Let G be a 2-edge-connected graph, and let e be an edge of G. Then G has a cc-minor H that is the **parallel connection**, with basepoint e, of a collection of cycles containing e.

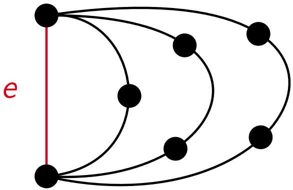

Figure: parallel connection of three cycles with basepoint e

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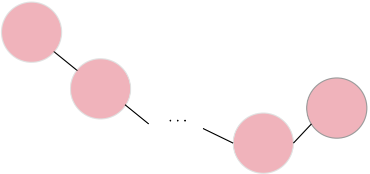
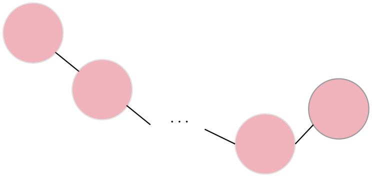


Case II

Proving the main result

Case I



Proving the main result

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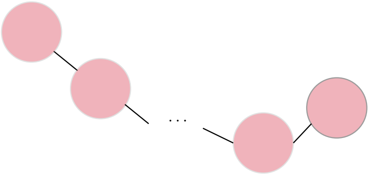
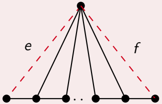
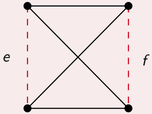


Lemma

Let G be a simple 3-connected graph, and let e and f be two distinct edges. Then G has a cc-minor H containing e and f such that the simplification of H is one of the following.





Proving the main result

Case II

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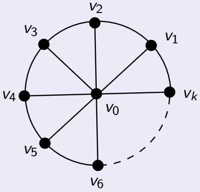
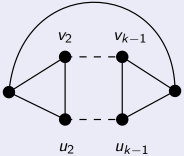
Case II

Theorem (Oporowski, Oxley, Thomas 1993)

A giant 3-connected graph contains a large subgraph isomorphic to a subdivision of one of W_k, V_k, and $K_{3,k}$.



k-spoke wheel W_k



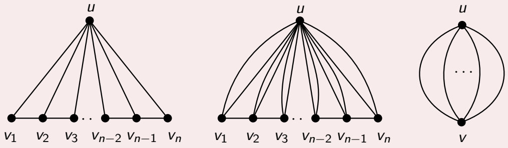
V_k

Proving the main result

Case II

Lemma

A giant 3-connected graph G contains a large fan-type graph as cc-minor.



$F_{1,1,\ldots,1}$ $F_{2,2,\ldots,2}$ B_n

Extension to regular matroids?

- Extend results of induced subgraphs and cc-minors to regular matroids using Seymour's decomposition theorem.

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Theorem

For giant 2-connected graph, the unavoidable cc-minors are parallel-path extension of large r-templates.