The Kővári–Sós–Turán theorem for GF(q)-representable matroids

Wayne Ge

Auburn University

Turán problems in graphs

For a fixed graph H with at least one edge, the extremal number

$$\mathrm{ex}(H; n) = \max\{|E(G)| : G \text{ is an } n\text{-vertex } H\text{-free graph}\}.$$

Here, H-free means that G has no subgraph isomorphic to H.

Turán problems in graphs

For a fixed graph H with at least one edge, the extremal number

$$\mathrm{ex}(H; n) = \max\{|E(G)| : G \text{ is an } n\text{-vertex } H\text{-free graph}\}.$$

Here, H-free means that G has no subgraph isomorphic to H.

In K_n, how many red edges force a red copy of H?

Dense cases

Mantel's and Turán's theorems

$$\mathrm{ex}(K_k; n) \leq \frac{k-2}{k-1} \cdot \frac{n^2}{2} \qquad (k \geq 2).$$

Dense cases

Mantel's and Turán's theorems

$$\mathrm{ex}(K_k; n) \leq \frac{k-2}{k-1} \cdot \frac{n^2}{2} \qquad (k \geq 2).$$

Erdős–Stone–Simonovits

$$\mathrm{ex}(H; n) = \left(1 - \frac{1}{\chi(H)-1} + o(1)\right)\binom{n}{2}.$$

Here $\chi(H)$ is the chromatic number.

Dense cases

Mantel's and Turán's theorems

$$\mathrm{ex}(K_k; n) \leq \frac{k-2}{k-1} \cdot \frac{n^2}{2} \qquad (k \geq 2).$$

Erdős–Stone–Simonovits

$$\mathrm{ex}(H; n) = \left(1 - \frac{1}{\chi(H)-1} + o(1)\right)\binom{n}{2}.$$

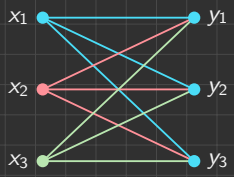
Here $\chi(H)$ is the chromatic number.

For bipartite H, this only gives $\mathrm{ex}(H; n) = o(n^2)$.

The bipartite Turán problem

Kővári–Sós–Turán (1954)

$$\mathrm{ex}(K_{s,t}; n) = O_{s,t}\left(n^{2-1/s}\right), \qquad s \le t.$$

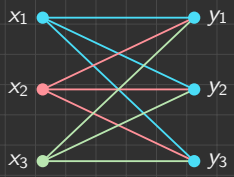


$K_{3,3}$

The bipartite Turán problem

Kővári–Sós–Turán (1954)

$$\mathrm{ex}(K_{s,t}; n) = O_{s,t}\left(n^{2-1/s}\right), \qquad s \le t.$$



$K_{3,3}$

In a $K_{s,t}$-free graph, a set of s vertices has at most $t-1$ common neighbors.

Count s-stars in two ways

An s-star is a pair (S, v), where $|S| = s$ and v is a common neighbor of S.

$$\#\{s\text{-stars}\} \leq \underbrace{(t-1)\binom{n}{s}}_{\text{count by the leaves}}.$$

Count s-stars in two ways

An s-star is a pair (S, v), where $|S| = s$ and v is a common neighbor of S.

$$\underbrace{\sum_{v \in V(G)} \binom{d(v)}{s}}_{\text{count by the center}} = \#\{s\text{-stars}\} \leq \underbrace{(t-1)\binom{n}{s}}_{\text{count by the leaves}}.$$

Count s-stars in two ways

An s-star is a pair (S, v), where $|S| = s$ and v is a common neighbor of S.

$$\underbrace{\sum_{v \in V(G)} \binom{d(v)}{s}}_{\text{count by the center}} = \#\{s\text{-stars}\} \leq \underbrace{(t-1)\binom{n}{s}}_{\text{count by the leaves}}.$$

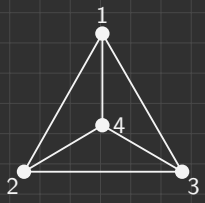
By convexity, with $e = |E(G)|$,

$$n\binom{2e/n}{s} \leq (t-1)\binom{n}{s}, \quad \text{so} \quad n\left(\frac{2e}{n}\right)^s = O_{s,t}(n^s).$$

Hence

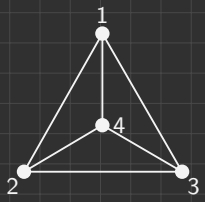
$$e = O_{s,t}\left(n^{2-1/s}\right).$$

From a graph to a matroid



K_4

From a graph to a matroid



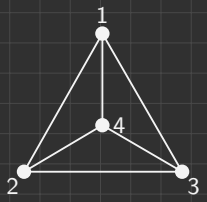
K_4

	12	13	14	23	24	34
1	1	1	1	0	0	0
2	1	0	0	1	1	0
3	0	1	0	1	0	1
4	0	0	1	0	1	1

Incidence matrix over $\mathbb{F}_2$

A circuit is a minimally linearly dependent set of columns.
In $M(G)$, the circuits are exactly the graph's cycles.

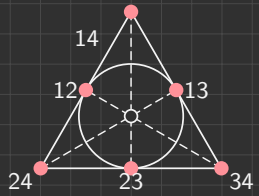
From a graph to a matroid



K_4

	12	13	14	23	24	34
1	1	1	1	0	0	0
2	1	0	0	1	1	0
3	0	1	0	1	0	1
4	0	0	1	0	1	1

Incidence matrix over $\mathbb{F}_2$



$M(K_4) \subseteq \mathrm{PG}(2,2)$

A circuit is a minimally linearly dependent set of columns.

In $M(G)$, the circuits are exactly the graph's cycles.

A restriction selects a subset of the points (or matrix columns).

Turán problems in matroids

Fix a finite field $\mathbb{F}_q$ and a simple GF(q)-representable matroid H.

$\mathrm{ex}_q(H; n)$ is the maximum number of elements in a rank-n simple GF(q)-representable matroid with no H-restriction.

Turán problems in matroids

Fix a finite field $\mathbb{F}_q$ and a simple GF(q)-representable matroid H.

$\mathrm{ex}_q(H; n)$ is the maximum number of elements in a rank-n simple GF(q)-representable matroid with no H-restriction.

The exclusion is as a restriction.

A simple representable matroid is a set of projective points:

$$E(M) \subseteq \mathrm{PG}(n-1, q), \qquad |\mathrm{PG}(n-1, q)| = \frac{q^n - 1}{q - 1}.$$

The graph–matroid dictionary

Graphs	GF(q)-representable matroids
Number of vertices	Rank
Number of edges	Number of elements
Subgraph	Restriction
Complete graph	Projective geometry
Chromatic number	Critical exponent
Bipartite forbidden graph	Affine forbidden matroid
Complete bipartite graph	Affine geometry

Bose–Burton (1966): matroidal Turán

For $n \geq m \geq 2$,

$$\mathrm{ex}_q(\mathrm{PG}(m-1,q);n) = \underbrace{\frac{q^n-1}{q-1}}_{|\mathrm{PG}(n-1,q)|} - \underbrace{\frac{q^{n-m+1}-1}{q-1}}_{|\mathrm{PG}(n-m,q)|}.$$

An extremal example deletes a rank-$(n-m+1)$ flat from $\mathrm{PG}(n-1,q)$.

Complete graphs $\longleftrightarrow$ Projective geometries

Geelen–Nelson (2015): matroidal Erdős–Stone

For fixed nonempty simple GF(q)-representable H,

$$\mathrm{ex}_q(H;n) = \left(1 - q^{1-c(H;q)} + o(1)\right) \frac{q^n - 1}{q - 1}.$$

$c(H;q)$ is the critical exponent: the matroid analogue of the chromatic number.

Geelen–Nelson (2015): matroidal Erdős–Stone

For fixed nonempty simple GF(q)-representable H,

$$\mathrm{ex}_q(H;n) = \left(1 - q^{1-c(H;q)} + o(1)\right)\frac{q^n - 1}{q-1}.$$

$c(H;q)$ is the critical exponent: the matroid analogue of the chromatic number.

$$c(H;q) = 1 \iff H \text{ is affine.}$$

In the affine case, the theorem only gives $\mathrm{ex}_q(H;n) = o(q^n)$.

What happens when the forbidden matroid is affine?

In sufficiently large hosts:

- complete bipartite graphs contain every fixed bipartite graph;
- affine geometries contain every fixed affine GF(q)-representable matroid.

What happens when the forbidden matroid is affine?

In sufficiently large hosts:

- complete bipartite graphs contain every fixed bipartite graph;
- affine geometries contain every fixed affine GF(q)-representable matroid.

Bonin–Qin (2000): the binary case

$$\mathrm{ex}_2(\mathrm{AG}(m-1,2);n) < 2^{a_m n+1}, \qquad a_m = 1-2^{-(m-2)}, \qquad n \geq m \geq 3.$$

Every fixed binary affine matroid inherits an exponential saving.

What happens when the forbidden matroid is affine?

In sufficiently large hosts:

- complete bipartite graphs contain every fixed bipartite graph;
- affine geometries contain every fixed affine GF(q)-representable matroid.

Bonin–Qin (2000): the binary case

$$\mathrm{ex}_2(\mathrm{AG}(m-1,2);n) < 2^{a_m n+1}, \qquad a_m = 1-2^{-(m-2)}, \qquad n \geq m \geq 3.$$

Every fixed binary affine matroid inherits an exponential saving.

A natural question

For $q \geq 3$ and every fixed affine H, is there $\delta > 0$ such that

$$\mathrm{ex}_q(H;n) = O_{q,H}\left(q^{(1-\delta)n}\right)?$$

Two results over $\mathbb{F}_3$

Ellenberg–Gijswijt (2017): cap sets

$$\mathrm{ex}_3(M(K_3); n) = O\left(3^{(1-\delta)n}\right), \qquad \delta > 0.$$

Two results over $\mathbb{F}_3$

Ellenberg–Gijswijt (2017): cap sets

$$\mathrm{ex}_3(M(K_3); n) = O\left(3^{(1-\delta)n}\right), \qquad \delta > 0.$$

Fox–Pham (2019): higher-dimensional affine geometries

$$\mathrm{ex}_3(\mathrm{AG}(m-1, 3); n) = O_m\left(3^{(1-\delta_m)n}\right), \qquad \delta_m > 0$$

for each fixed $m \geq 2$.

Remark. Thanks to Bryce Frederickson for introducing these results.

Where we stand

Graphs	Matroids
Mantel; Turán	Bose–Burton
Erdős–Stone	Geelen–Nelson
Kővári–Sós–Turán	Bonin–Qin; Fox–Pham

Where we stand

Graphs	Matroids
Mantel; Turán	Bose–Burton
Erdős–Stone	Geelen–Nelson
Kővári–Sós–Turán	Bonin–Qin; Fox–Pham

$M(K_{s,t})$ is affine over every finite field.

$$\mathrm{ex}_q(M(K_{s,t}); n) = ?$$

The matroidal Kővári–Sós–Turán bound

For every prime power q and fixed $2 \le s \le t$,

$$\boxed{\mathrm{ex}_q(M(K_{s,t}); n) = O_{q,s,t}\left(q^{(1-1/s)n}\right).}$$

$M(K_{s,t})$ is affine over every $\mathrm{GF}(q)$.

The matroidal Kővári–Sós–Turán bound

For every prime power q and fixed $2 \le s \le t$,

$$\boxed{\mathrm{ex}_q(M(K_{s,t}); n) = O_{q,s,t}\left(q^{(1-1/s)n}\right).}$$

$M(K_{s,t})$ is affine over every $\mathrm{GF}(q)$.

Compare, for $q = 2$, Bonin–Qin:

$$\mathrm{ex}_2(M(K_{s,t}); n) = O\left(2^{(1-2^{-(s+t-3)})n}\right).$$

The saving improves from $2^{-(s+t-3)}$ to $1/s$.

Proof sketch: reproduce the s-star count.

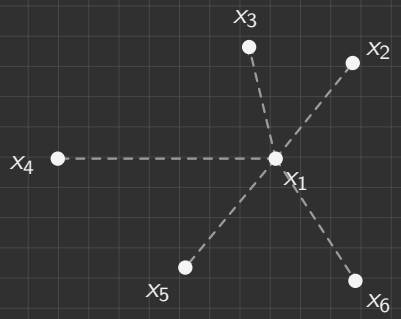
How can we translate the proof?

Graph proof	Matroid setting
Edges	Matroid elements
Vertices	???
Common neighbors	???

How can we translate the proof?

Graph proof	Matroid setting
Edges	Matroid elements
Vertices	An affinely independent s-tuple X of vectors
Common neighbors	Translates y with $X + y \subseteq A$

A set of s vertices



Fix $\mathbb{F}_q$;
$E(M) = A \subseteq \mathbb{F}_q^n$, a set of vectors.

A "set of s vertices" is an ordered tuple

$$X = (x_1, x_2, \ldots, x_s) \in (\mathbb{F}_q^n)^s$$

of affinely independent vectors, i.e.

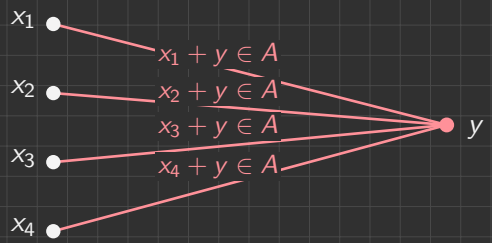
$x_2 - x_1, \ldots, x_s - x_1$ are linearly independent.

The vectors x_i need not lie in A.

Common neighbors

$y \in \mathbb{F}_q^n$ is a common neighbor of $X = (x_1, \ldots, x_s)$ if $x_i + y \in A$ for every i.

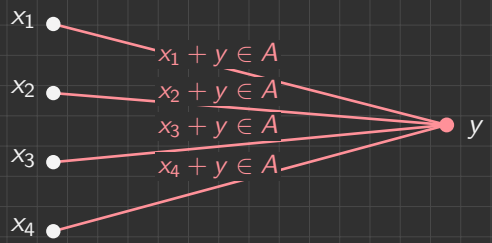
Write $N(X)$ for the set of common neighbors.



Common neighbors

$y \in \mathbb{F}_q^n$ is a common neighbor of $X = (x_1, \ldots, x_s)$ if $x_i + y \in A$ for every i.

Write $N(X)$ for the set of common neighbors.



Not every common neighbor gives an $M(K_{1,s})$.

When does a common neighbor give $M(K_{1,s})$?

Put $v_i = x_i - x_1$ and

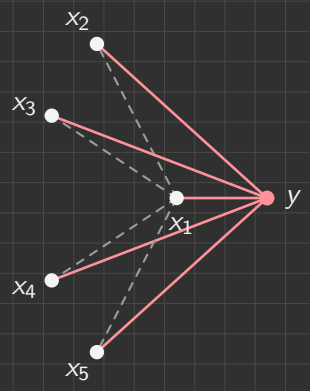
$$W = \operatorname{span}\{x_2 - x_1, x_3 - x_1, \ldots, x_s - x_1\}, \qquad \dim W = s - 1.$$

Since $x_i + y = (x_1 + y) + v_i$, the set

$$\{x_1 + y, \ldots, x_s + y\}$$

is independent, i.e. an $M(K_{1,s})$,

exactly when $x_1 + y \notin W$.



Many neighbors $\Rightarrow$ a forbidden $M(K_{s,t})$

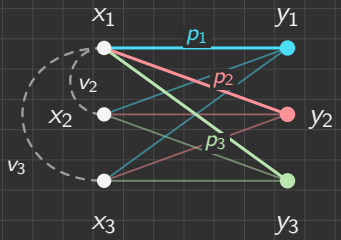
If there exist $y_1, \dots, y_t \in N(X)$ such that

the cosets $x_1 + y_1 + W, \dots, x_1 + y_t + W$ are independent,

put $v_i = x_i - x_1$ and $p_j = x_1 + y_j \quad (v_1 = 0)$.

Then $v_2, \dots, v_s, p_1, \dots, p_t$ are linearly independent.

$$x_i + y_j = v_i + p_j \in A$$



	y_1	y_2	y_3
x_1	p_1	p_2	p_3
x_2	$v_2 + p_1$	$v_2 + p_2$	$v_2 + p_3$
x_3	$v_3 + p_1$	$v_3 + p_2$	$v_3 + p_3$

A constant bound on common neighbors

Since M has no $M(K_{s,t})$-restriction,

$$\dim \operatorname{span}\{x_1 + y + W : y \in N(X)\} \leq t - 1.$$

$$\boxed{|N(X)| \leq q^{s-1} \cdot q^{t-1} = q^{s+t-2}.}$$

Double count the pairs (X, y) with $y \in N(X)$

Upper bound: count by the leaves X

$$\#(X, y) = \sum_X |N(X)| \leq q^{ns} \cdot q^{s+t-2}.$$

Double count the pairs (X, y) with $y \in N(X)$

Upper bound: count by the leaves X

$$\#(X, y) = \sum_X |N(X)| \le q^{ns} \cdot q^{s+t-2}.$$

Lower bound: count by the center y

For every affinely independent s-tuple $(a_1, \dots, a_s) \in A^s$ and every $y \in \mathbb{F}_q^n$,

$$((a_1 - y, \dots, a_s - y), y) \text{ is counted.}$$

Translation preserves affine independence, so

$$\#(X, y) = \left|\{\text{affinely independent } s\text{-tuples in } A^s\}\right| \cdot q^n.$$

For fixed q, s and sufficiently large $|A|$, there are $\Theta_{q,s}(|A|^s)$ such tuples.

Compare the two counts

By the leaves X $\#(X, y) \leq q^{ns} \cdot q^{s+t-2}$

By the center y $\#(X, y) \geq c\,|A|^s q^n$

Combined $|A|^s \leq C_{q,s,t} q^{(s-1)n}$

$$|E(M)| = |A| = O_{q,s,t}\left(q^{(1-1/s)n}\right)$$

Binary Sidon sets

Binary Sidon set $S \subseteq \mathbb{F}_2^n$: no distinct $a, b, c, d \in S$ with $a + b = c + d$, i.e. no $M(C_4) = M(K_{2,2})$ (for $0 \notin S$).

Sidon sets with $|S| = \Omega(2^{n/2})$ exist, so

$$\boxed{\mathrm{ex}_2(M(K_{2,t}); n) = \Theta_t(2^{n/2})} \qquad (t \geq 2).$$

Beyond complete bipartite graphs

Let H be bipartite with a cycle and degree at most s on one side.

Alon–Krivelevich–Sudakov (2003)

$$\mathrm{ex}(H; n) = O_H(n^{2-1/s}), \quad s \geq 2.$$

Frederickson–Yepremyan (2026)

$$\mathrm{ex}_q(M(H); n) = O_{q,H}(q^{(1-1/s)n}).$$

A Bondy–Simonovits analogue

Bondy–Simonovits (1974)

$$\mathrm{ex}(C_{2k}; n) = O_k(n^{1+1/k}), \quad k \geq 2.$$

Frederickson–Yepremyan (2026)

For fixed $k \geq 2$,

$$\mathrm{ex}_q(M(C_{2k}); n) = O_{q,k}(q^{n/k}).$$

For $q = 2$, $\mathrm{ex}_2(M(C_{2k}); n) = \Theta_k(2^{n/k})$.

What will be good to know?

Forbidding complete p-partite graphs, for some $p \leq q$

$M(K_{a_1,\ldots,a_p})$ is affine over $\mathrm{GF}(q)$.

Geelen–Nelson: affine targets

$$\mathrm{ex}_q(M(K_{a_1,\ldots,a_p}); n) = o(q^n).$$

Can we always force exponential loss?

Recall cap sets

Ellenberg–Gijswijt (2017), $q \geq 3$

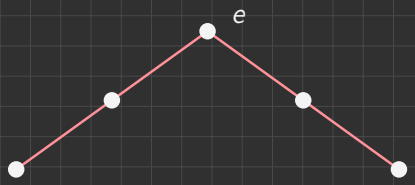
$$C_q(n) = \mathrm{ex}_q(M(K_3); n) = O(q^{(1-\delta_q)n}), \qquad \delta_q > 0.$$

Triangles and $K_{1,1,t}$

In $\mathbb{F}_q$, start with $K_{1,1,1}$ and $K_{1,1,2}$.



$M(K_{1,1,1}) = M(K_3)$



$M(K_{1,1,2}) = M(K_4^-)$

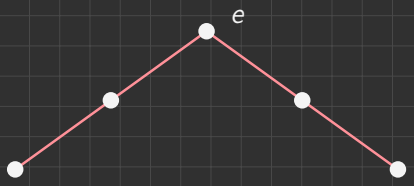
A $K_{1,1,2}$-free matroid has disjoint long lines.

Triangles and $K_{1,1,t}$

In $\mathbb{F}_q$, start with $K_{1,1,1}$ and $K_{1,1,2}$.



$M(K_{1,1,1}) = M(K_3)$



$M(K_{1,1,2}) = M(K_4^-)$

A $K_{1,1,2}$-free matroid has disjoint long lines.

$$C_q(n) \leq \mathrm{ex}_q(M(K_{1,1,2}); n) \leq (q+1)/2\, C_q(n).$$

Recreational problem

For fixed q and $t \geq 2$,

can we keep the same cap-set growth?

$$\mathrm{ex}_q(M(K_{1,1,t}); n) = O_{q,t}(C_q(n)) \ ?$$

Recreational problem

For fixed q and $t \geq 2$,

can we keep the same cap-set growth?

$$\mathrm{ex}_q(M(K_{1,1,t}); n) = O_{q,t}(C_q(n)) \ ?$$

What changes when the two singleton parts grow?

$$K_{1,1,t} \longrightarrow K_{a,b,c}$$

From $K_{1,1,t}$ to $K_{a,b,c}$

Let $1 \le a \le b \le c$.

A consequence of Fox–Pham (2019)

$$\mathrm{ex}_3(M(K_{a,b,c}); n) = O_{a,b,c}(3^{(1-\delta)n}).$$

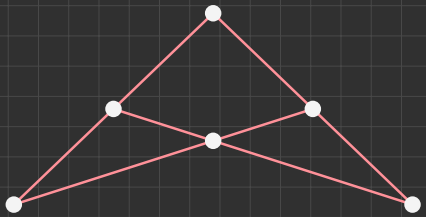
for some $\delta = \delta(a, b, c) > 0$.

How does the best δ depend on a, b, c?

Can the saving be independent of c?

First targets over $\mathbb{F}_4$

Both targets are affine over GF(4).



$M(K_4)$



$U_{2,4}$

For either target N, can we force exponential loss?

$$\mathrm{ex}_4(N; n) = O_N(4^{(1-\delta)n}), \quad \delta > 0 \ ?$$