Oriented diameter of near planar triangulations

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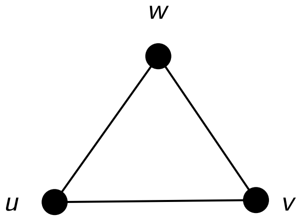
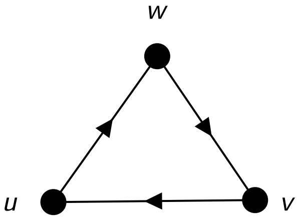
Digraph

- Let $G = (V, E)$ be an undirected graph
- An **orientation** of G is a digraph...
  "Turning the streets in a town into one-way streets."





- uv path $\neq$ vu path in digraphs

Diameter

- Given a digraph D, the **distance function**

$$d(u, v) := \text{length of the shortest } uv \text{ path.}$$

- **Diameter** of a digraph D

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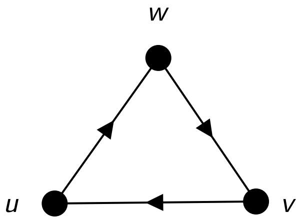
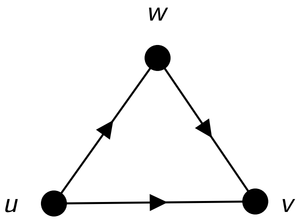
- A digraph D is **strongly connected** if for all ordered pairs $\{u, v\} \subseteq V(D)$, there exists a uv path.





not strongly connected

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Theorem (Robbins 1939)

Any connected bridgeless graph G admits a strongly connected orientation.

Now we only consider connected, bridgeless graphs.

Oriented diameter of a graph

- The **oriented diameter** of a undirected graph G is the minimum diameter over the strongly connected orientations of G,

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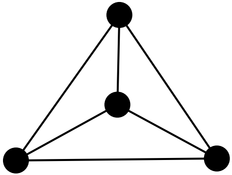
e.g. $\overrightarrow{diam}(K_4) = 3;\qquad \overrightarrow{diam}(K_n) = 2 \textit{ for all } n \geq 3, n \neq 4.$



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In general, computing the oriented diameter of a given graph G is hard.

Theorem (Chvátal, Thomassen 1978)

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Theorem (Bau, Dankelmann and Henning 2005)

The only planar graphs with an orientation of diameter 2 are the triangle and the octahedron.

But for arbitrary planar graph G, computing $\overrightarrow{\text{diam}}(G)$ is still hard.

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Theorem (Babu, Benson, Rajendraprasad and Veka 2021)

Let G *be a graph with* $diam(G) = d$*. Then*

$$\overrightarrow{diam}(G) \leq 1.373d^2 + 6.971d - 1.$$

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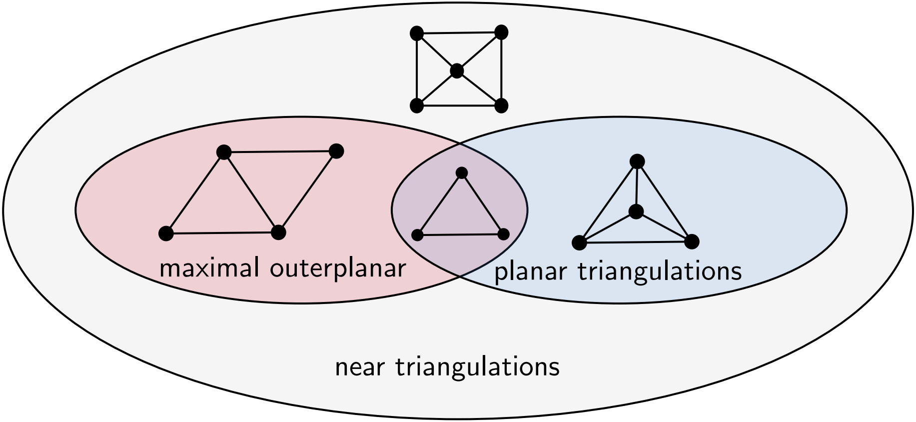
- An **outerplanar graph** is a planar graph such that every vertex lies on the boundary of the outer face.
- A **maximal outerplanar graph** is a outerplanar graph such that every face except possibly the outer face is a triangle.
- A **near triangulation** is a planar graph such that every face except possibly the outer face is a triangle.

A Venn diagram



Previous results

Theorem (Wang, Chen, Dankelmann, Guo, Surmacs and Volkmann 2021)

Let G be an n-vertex maximal outerplanar graph with $n \geq 3$. Then $\overrightarrow{diam}(G) \leq \lceil \frac{n}{2} \rceil$ and this bound is tight.

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Unless, $G \in \{K_4^-, G_6^1, G_6^2, G_8^1\}$.

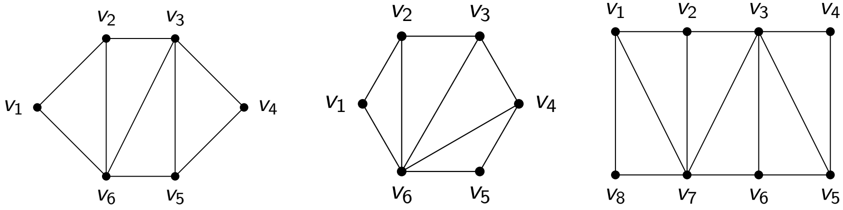


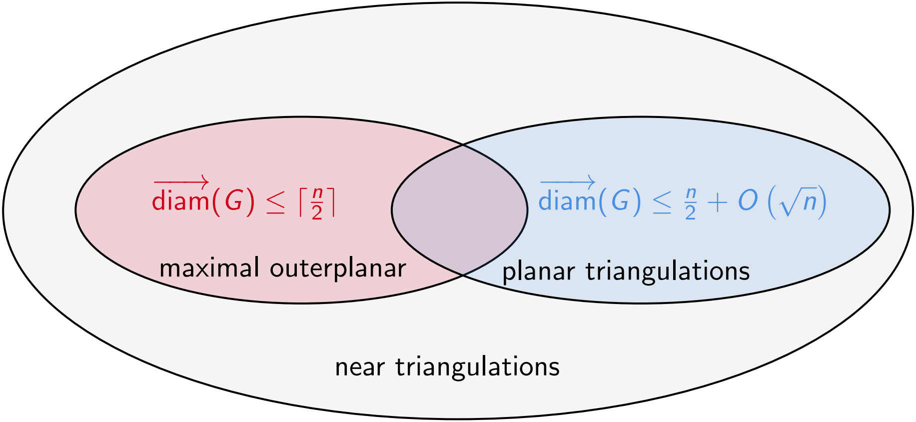
Figure: G_6^1, G_6^2, and G_8^1 .

Previous results

Theorem (Mondal, Parthiban and Rajasingh 2023)

Let G be an n-vertex planar triangulation. Then $\overrightarrow{diam}(G) \leq \frac{n}{2} + O(\sqrt{n})$.

Venn diagram II



Our result

Theorem (G., Liu and Wang 2023+)

Let G be an n-vertex 2-connected near triangulation. Then $\overrightarrow{diam}(G) \leq \lceil \frac{n}{2} \rceil$ and the bound is tight.

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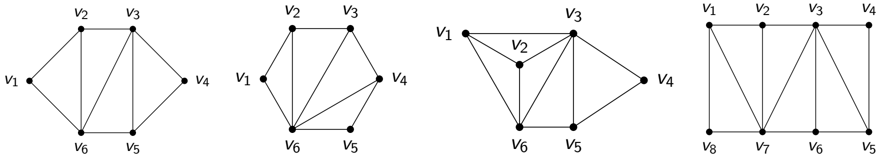
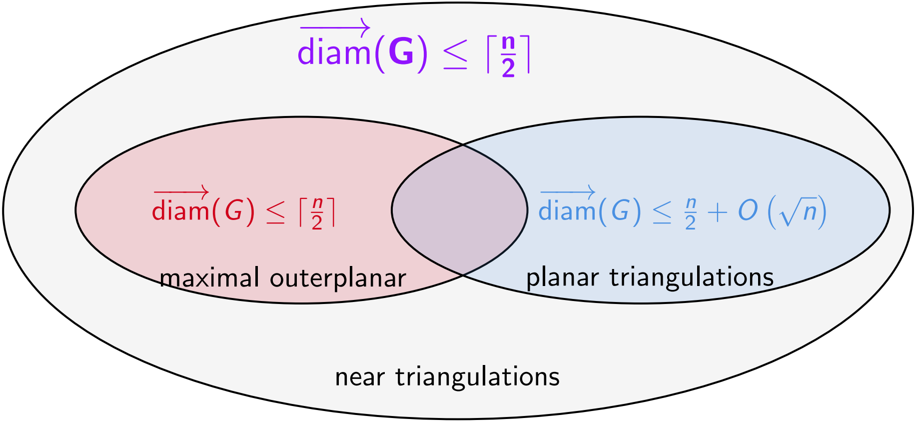


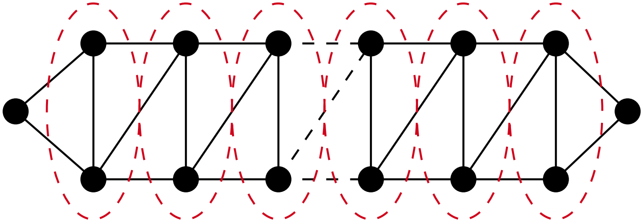
Figure: G_6^1, G_6^2, G_6^3, and G_8^1.

Venn diagram III

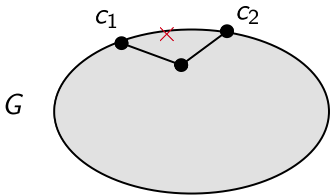


Tightness

Our bound is tight for near planar triangulations.



Proof sketch



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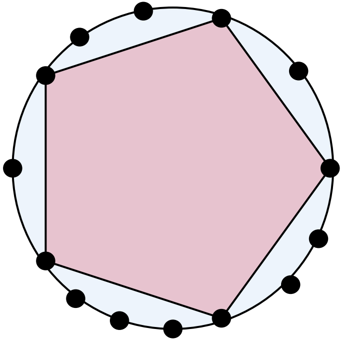

- Note every outerplanar graph has at least two degree-2 vertices.

Proof sketch

Note: Having lots of degree-2 vertices is impossible.

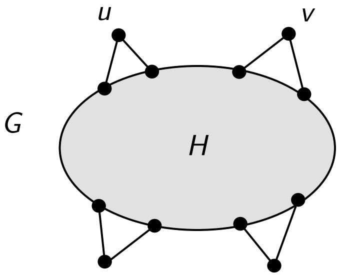

Figure: Four degree-2 vertices

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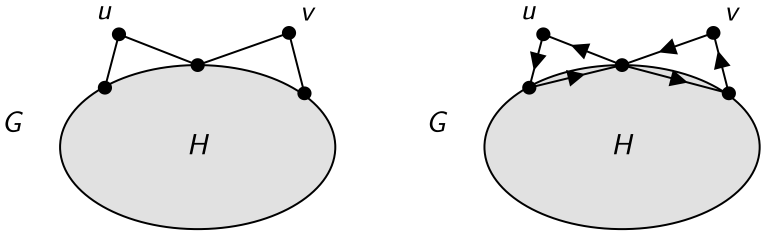
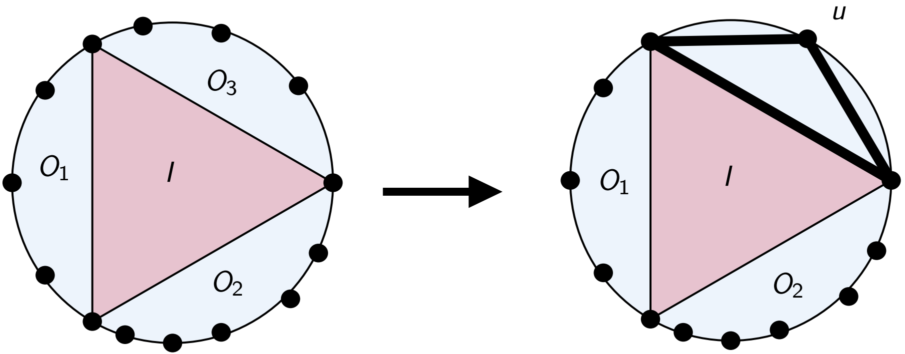


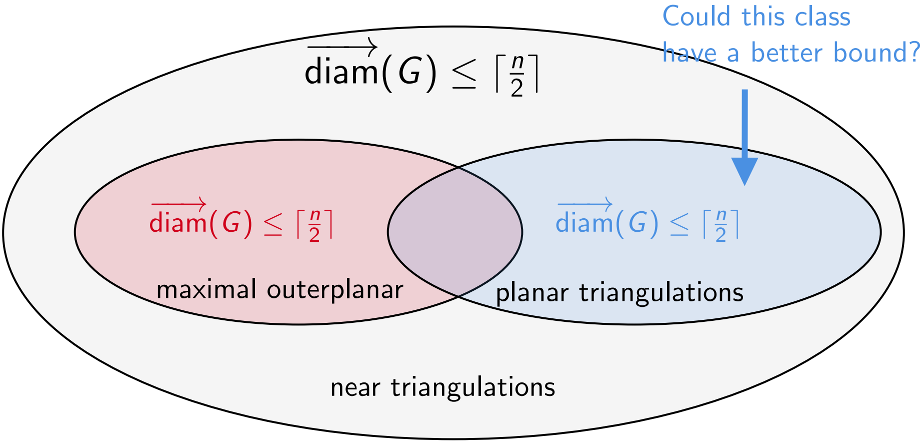
Figure: Two adjacent degree-2 vertices

Proof sketch



At most one blue part is a triangle.

Future direction



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Conjecture

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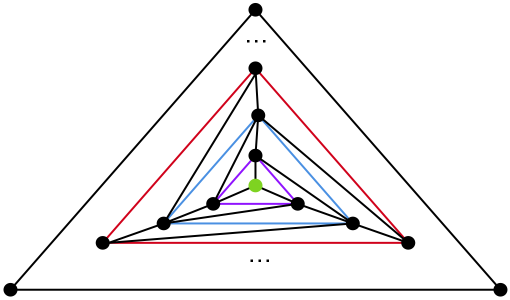
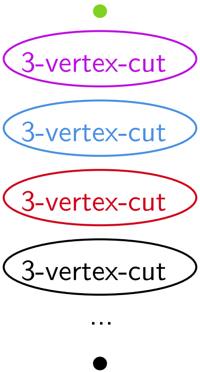
If such an $n/3$ upper bound is obtained, it will be tight for planar triangulations.





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Q: With other conditions, can we bound $\overrightarrow{diam}(G)$ by $f(r)$ or $f(d)$ where f being linear?

Thank you!