Extremal connectivity in graphs and matroids

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NORTH HUB 6
STATION 1
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different levels of connectivity

Definition

A k-connected graph G is **minimally** k**-connected** if $G \setminus e$ is not k-connected for all $e \in E(G)$.

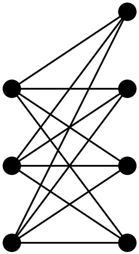

Figure: minimally 3-connected

different levels of connectivity

Definition

A k-connected graph G is **critically k-connected** if $G - v$ is not k-connected for all $v \in V(G)$.

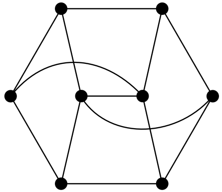

Figure: critically 3-connected

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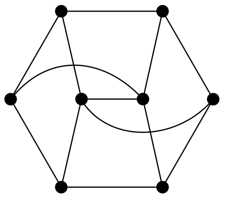

Figure: critically 3-connected

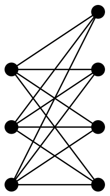

Figure: not critically 3-connected

different levels of connectivity

Definition

A graph G is **uniformly** k**-connected** if, for every pair of distinct vertices, there are exactly k internally disjoint paths connecting them.

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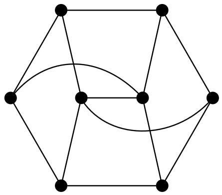
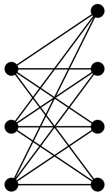


Figure: not uniformly 3-connected



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Inclusion lemma

For all integers k exceeding one and all graphs G,

(i) if G is uniformly k-connected, then G is super-minimally k-connected, and

(ii) if G is super-minimally k-connected, then G is both minimally and critically k-connected.

Moreover, for all $k \geq 3$, neither of the converses holds.

Hierarchy of 3-connected classes

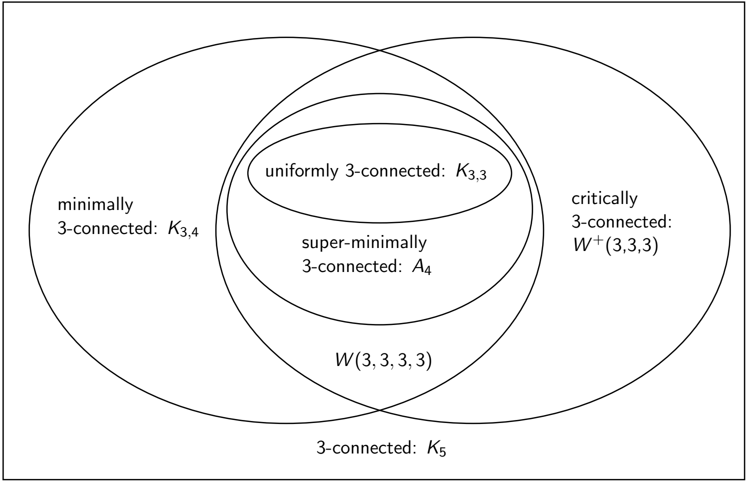
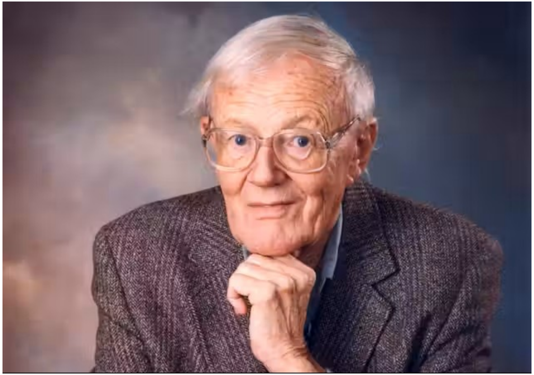


Figure: When $k \geq 3$, uniformly $\subset$ super-minimally $\subset$ minimally and critically

What about matroids?



"If a theorem about graphs can be expressed in terms of edges and circuits only, it probably exemplifies a more general theorem about matroids."–W. T. Tutte

extremal connectivity in matroids

Definition

A k-connected matroid M is **minimally k-connected** if $M \backslash e$ is not k-connected for every $e \in E(M)$.

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Example

For each $k \geq 2$, the uniform matroid $U_{r,r+k-1}$ is minimally k-connected for every $r \geq k+1$.

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For each $k \geq 2$, the uniform matroid $U_{r,r+k-1}$ is minimally k-connected for every $r \geq k+1$.
$\lambda(U_{r,n}) = n - r + 1$ when $n \leq 2r - 2$ [Inukai and Weinberg (1978)] (see Oxley 8.6.3).

extremal connectivity in matroids

Definition

A k-connected matroid M is **super-minimally k-connected** if M has no proper k-connected restriction of size at least $2k - 2$.

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Example

When $k = 2, 3$, the cycle matroid of every super-minimally k-connected graph is super-minimally k-connected.

super-minimally k-connected $\implies$ minimally k-connected?

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Example

$U_{1,3}$ is

1. 3-connected,

super-minimally k-connected $\implies$ minimally k-connected?

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$U_{1,3}$ is

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$U_{1,3}$ is

1. 3-connected,
2. super-minimally 3-connected, but
3. not minimally 3-connected ($U_{1,2}$ is 3-connected).

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Proposition

Let $k \geq 2$ be an integer and let M be a matroid.

(i) If $|E(M)| \leq 2k - 2$, then M is super-minimally k-connected if and only if M is k-connected.

(ii) If M is super-minimally k-connected and $|E(M)| > 2k - 2$, then M is minimally k-connected.

small super-minimally 3-connected matroids

Number n of elements	**3-connected n-element matroids**
1	$U_{0,1}$, $U_{1,1}$
2	$U_{1,2}$
3	$U_{1,3}$, $U_{2,3}$
4	$U_{2,4}$

Table: All 3-connected matroids on at most four elements.

density of minimally 2- and 3-connected graphs

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Theorem (Dirac 1967)

If G is a minimally 2-connected graph with $|V(G)| \geq 4$, then

$$|E(G)| \leq 2|V(G)| - 4.$$

Moreover, equality holds if and only if G is isomorphic to $K_{2,|V(G)|-2}$.

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Theorem (Halin 1969)

If G is a minimally 3-connected graph with $|V(G)| \geq 7$, then

$$|E(G)| \leq 3|V(G)| - 9.$$

Moreover, if $|V(G)| \geq 8$, equality holds if and only if G is isomorphic to $K_{3,|V(G)|-3}$.

density of minimally 2- and 3-connected matroids

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Theorem (Murty 1974)

If M is a minimally 2-connected matroid with $r(M) \geq 3$, then

$$|E(M)| \leq 2r(M) - 2.$$

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Theorem (Oxley 1981)

If M is a minimally 3-connected matroid with $r(M) \geq 3$, then

$$|E(M)| \leq \begin{cases} 2r(M), & \text{if} \quad r(M) \leq 6; \\ 3r(M) - 6, & \text{if} \quad r(M) \geq 7. \end{cases}$$

What about super-minimally 2- and 3-connected matroids?

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Proposition

A matroid M is super-minimally 2-connected if and only if M is isomorphic to $U_{1,1}$, or $U_{r,r+1}$ for some integer $r \geq 0$. In particular,

$$|E(M)| \leq r(M) + 1.$$

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Theorem (G., Oxley 2026+)

Let M be a super-minimally 3-connected matroid with at least four elements. Then

$$|E(M)| \leq 2r(M).$$

Moreover, equality holds if and only if $M \cong U_{2,4}$, or $M \cong M(\mathcal{W}_k)$ or $\mathcal{W}^k$ for some $k \geq 3$.

reduction theorems

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Theorem (Halin 1969; Dawes 1986)

Suppose G is a minimally 3-connected graph that is not K_4. Then G has

(I) *an edge e such that $\mathrm{co}(G \backslash e)$ is minimally 3-connected, or*

(II) *a degree-3 vertex v such that $G - v$ is minimally 3-connected.*

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- A matroid analogue was proved by Oxley (1981).

Bixby's Lemma

Lemma (Bixby 1982)

Let e be an element of a 3-connected matroid M. Then $\mathrm{si}(M/e)$ is 3-connected or $\mathrm{co}(M \setminus e)$ is 3-connected.

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Lemma

Let e be an element of a super-minimally 3-connected matroid M with at least five elements. Then $\mathrm{si}(M/e)$ is 3-connected or $\mathrm{co}(M \backslash e)$ is super-minimally 3-connected.

An application of the previous lemma

Reduction Lemma

Suppose M is a super-minimally 3-connected matroid with at least seven elements and $\{e, f, g\}$ is a triangle of M. Then M has an element x of $\{e, f, g\}$ such that $\mathrm{co}(M\backslash x)$ is super-minimally 3-connected.

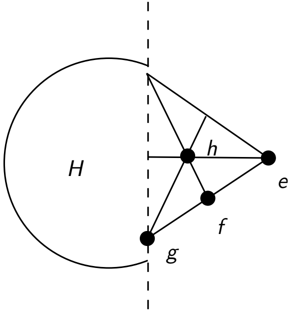

Figure: A triangle always has an element x such that $\mathrm{si}(M/x)$ is not 3-connected.

What if M has no triangle?

Fragile graphs

Definition

A graph G is **fragile** if G has no 3-connected subgraph.

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Proposition (Bonnet, Feghali, Nguyen, Scott, Seymour, Thomassé, Trotignon 2025)

If G is a fragile graph, then

$$|E(G)| \leq 2.5|V(G)| - 5.$$

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If G is a fragile graph, then

$$|E(G)| \leq 2.5|V(G)| - 5.$$

If G is a fragile graph of girth at least four, then

$$|E(G)| \leq 2|V(G)| - 4.$$

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Lemma

If M is a brittle matroid, then

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If M is a triangle-free brittle matroid, then

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If M is a triangle-free brittle matroid, then

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Corollary

If M is a triangle-free super-minimally 3-connected matroid with at least four elements, then

$$|E(M)| \leq 2r(M) - 1.$$

Matroids attaining the extremal density

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Lemma (Growing wheels and whirls)

Suppose M is a minimally 3-connected matroid such that $M\backslash x/y$ is isomorphic to $M(\mathcal{W}_k)$ or $\mathcal{W}^k$ for some $k \geq 2$. Then M is isomorphic to $M(\mathcal{W}_{k+1})$ or $\mathcal{W}^{k+1}$.

Proof strategy map

Suppose M is a minimal counterexample.

Does M have a triangle?

Yes

There is an element e such that $\mathrm{co}(M\backslash e)$ is a smaller counterexample. ※

No

Then M is triangle-free and $M\backslash e$ is brittle for every e. ※

Extremal cases: If $|E(M)| = 2r(M)$, then M has a triangle, and there are elements e, f such that $M\backslash e/f$ is a wheel or a whirl.

Therefore, M is a wheel or a whirl.

application to graphs

Theorem (G. 2025+)

If G is a super-minimally 3-connected graph, then

$$|E(G)| \leq 2|V(G)| - 2.$$

Moreover, equality is attained if and only if G is a wheel with at least three spokes.

degree-3 vertices

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Theorem (Halin 1969)

A minimally 3-connected graph G has at least $\frac{2|V(G)|+6}{5}$ vertices of degree three.

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Theorem (Göring, Hofmann, and Streicher 2022)

A uniformly 3-connected graph G has at least $\frac{2|V(G)|+2}{3}$ vertices of degree three.

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What about matroids?

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Theorem (Oxley 1981)

Let G be minimally 3-connected graph. Then

$$v_3 \geq \begin{cases} \frac{2|V(G)|+7}{5} & \textit{for } \frac{3|V(G)|}{2} \leq |E(G)| < \frac{9|V(G)|-3}{5}, \\ \frac{|V(G)|-|E(G)|+3}{2} & \textit{for } \frac{9|V(G)|-3}{5} \leq |E(G)| \leq 3|V(G)|-6. \end{cases}$$

I: number of triads in a rank-r matroid

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Theorem (Lemos 2007)

Let M be a minimally 3-connected matroid with at least four elements. Then M has at least $\frac{r(M)+6}{4}$ triads.

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Can we improve it under a similar strengthening of minimal connectivity?

I: number of triads in a rank-r matroid

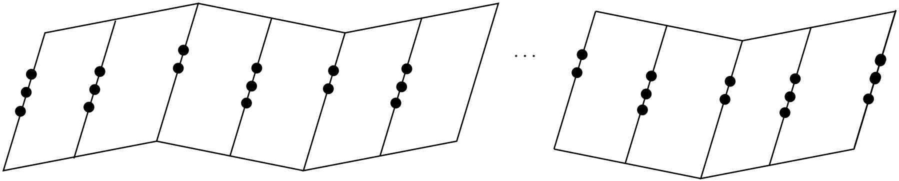

Figure: The dual of an extremal example.

II: number of elements belonging to triads

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Theorem (Lemos 2004)

Let M be a minimally 3-connected matroid with at least eight elements. Then the number of elements contained in at least one triad is at least $\frac{5|E(M)|+30}{9}$.

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Corollary

Let M be a super-minimally 3-connected matroid with at least eight elements. Then the number of elements contained in at least one triad is at least $\frac{5|E(M)|+30}{9}$.

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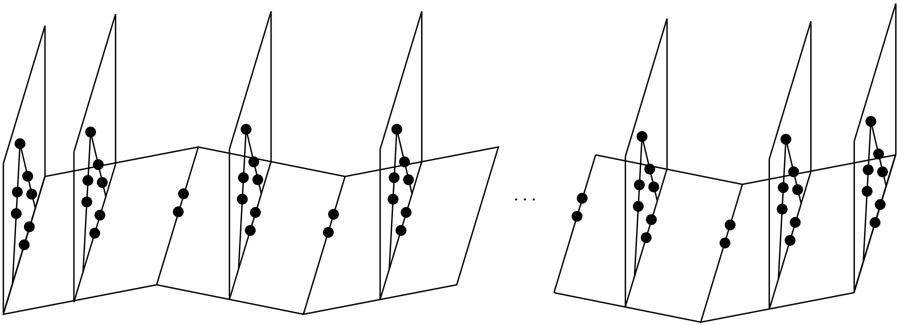

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Conjecture

Let M be a super-minimally 3-connected matroids such that $|E(M)| \geq 4$. Then there is an element e of M such that $\text{co}(M \backslash e)$ is super-minimally 3-connected.

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• A complete reduction/construction theorem for super-minimally 3-connected matroids/graphs.

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Let M be a super-minimally 3-connected matroids such that $|E(M)| \geq 4$. Then there is an element e of M such that $\mathrm{co}(M\backslash e)$ is super-minimally 3-connected.

Conjecture

Let G be a super-minimally 3-connected graph that $G \ncong K_4$. Then there is an edge e of G such that $\mathrm{co}(M\backslash e)$ is super-minimally 3-connected.

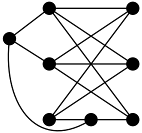

Figure: One example that our current reduction lemmas cannot be applied.

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An analogous problem for minimally 3-connected binary matroids was addressed by (Gomes, Lemos 2016). They improve the $\frac{5|E(M)|+30}{9}$ bound to $\frac{2|E(M)|+12}{3}$.

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A reduction theorem for minimally 3-connected binary matroids was proved by (Anderson, Wu 2008).

Some future directions

- Play the entire game again with **critically k-connected matroids**.

Definition

A matroid M is **critically k-connected** if M is k-connected and every hyperplane of M is not k-connected.

Summary

Class	Max $\lvert E(M)\rvert$	Min triads	Elements in triads
Minimally 3-conn.	$3r-6$	$\frac{r+6}{4}$	$\frac{5\lvert E(M)\rvert+30}{9}$
Super-min. 3-conn.	$2r$	$\frac{r+6}{4}$	$\frac{5\lvert E(M)\rvert+30}{9}$

Class	Min deg-3 vertices	Max edges
Minimally 3-conn.	$\frac{2n+6}{5}$	$3n-9$
Super-min. 3-conn.	$\frac{n+3}{2}$	$2n-2$
Uniformly 3-conn.	$\frac{2n+2}{3}$	$2n-2$

arXiv: 2603.11318

sm3c matroids

arXiv: 2510.06392

sm3c graphs