Turán problems for matroids representable over finite fields

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Turán problems in graphs

For a fixed graph H with at least one edge, the extremal number

$$\mathrm{ex}(H; n) = \max\{|E(G)| : G \text{ is an } n\text{-vertex } H\text{-free graph}\}.$$

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In K_n, how many red edges force a red copy of H?

Dense cases

Mantel's and Turán's theorems

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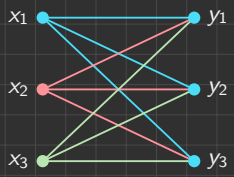
Here $\chi(H)$ is the chromatic number.

For bipartite H, this only gives $\mathrm{ex}(H; n) = o(n^2)$.

The bipartite Turán problem

Kővári–Sós–Turán (1954)

$$\mathrm{ex}(K_{s,t}; n) = O_{s,t}\left(n^{2-1/s}\right), \qquad s \le t.$$

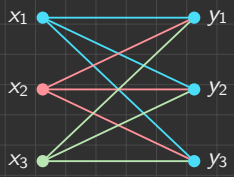


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In a $K_{s,t}$-free graph, a set of s vertices has at most $t-1$ common neighbors.

Count s-stars in two ways

An s-star is a pair (S, v), where $|S| = s$ and v is a common neighbor of S.

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By convexity, with $e = |E(G)|$,

$$n\binom{2e/n}{s} \leq (t-1)\binom{n}{s}, \quad \text{so} \quad n\left(\frac{2e}{n}\right)^s = O_{s,t}(n^s).$$

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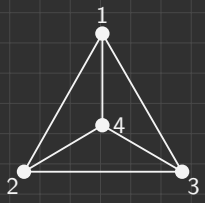
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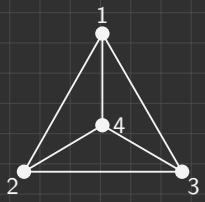
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From a graph to a matroid



K_4

From a graph to a matroid



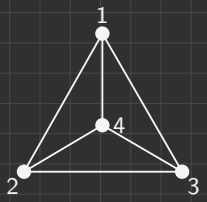
K_4

	12	13	14	23	24	34
1	1	1	1	0	0	0
2	1	0	0	1	1	0
3	0	1	0	1	0	1
4	0	0	1	0	1	1

Incidence matrix over $\mathbb{F}_2$

A circuit is a minimally linearly dependent set of columns.
In $M(G)$, the circuits are exactly the graph's cycles.

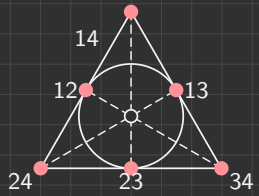
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Incidence matrix over $\mathbb{F}_2$



$M(K_4) \subseteq \mathrm{PG}(2,2)$

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In $M(G)$, the circuits are exactly the graph's cycles.

A restriction selects a subset of the points (or matrix columns).

Turán problems in matroids

Fix a finite field $\mathbb{F}_q$ and a simple GF(q)-representable matroid H.

$\mathrm{ex}_q(H; n)$ is the maximum number of elements in a rank-n simple GF(q)-representable matroid with no H-restriction.

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$\mathrm{ex}_q(H; n)$ is the maximum number of elements in a rank-n simple GF(q)-representable matroid with no H-restriction.

Excluding H as a restriction is the analogue of excluding a subgraph.

A simple representable matroid is a set of projective points:

$$E(M) \subseteq \mathrm{PG}(n-1, q), \qquad |\mathrm{PG}(n-1, q)| = \frac{q^n - 1}{q - 1}.$$

The graph–matroid dictionary

Graphs	GF(q)-representable matroids
Number of vertices	Rank
Number of edges	Number of elements
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Complete graph	Projective geometry

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Chromatic number	Critical exponent
Bipartite forbidden graph	Affine forbidden matroid
Complete bipartite graph	Affine geometry

Bose–Burton (1966): matroidal Turán

For $n \geq m \geq 2$,

$$\mathrm{ex}_q(\mathrm{PG}(m-1,q);n) = \underbrace{\frac{q^n-1}{q-1}}_{|\mathrm{PG}(n-1,q)|} - \underbrace{\frac{q^{n-m+1}-1}{q-1}}_{|\mathrm{PG}(n-m,q)|}.$$

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An extremal example deletes a rank-$(n-m+1)$ flat from $\mathrm{PG}(n-1,q)$.

Complete graphs $\longleftrightarrow$ Projective geometries

Geelen–Nelson (2015): matroidal Erdős–Stone

For fixed nonempty simple GF(q)-representable H,

$$\mathrm{ex}_q(H;n) = \left(1 - q^{1-c(H;q)} + o(1)\right) \frac{q^n - 1}{q - 1}.$$

$c(H;q)$ is the critical exponent: the matroid analogue of the chromatic number.

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$$c(H;q) = 1 \iff H \text{ is affine.}$$

In the affine case, the theorem only gives $\mathrm{ex}_q(H;n) = o(q^n)$.

What happens when the forbidden matroid is affine?

In sufficiently large hosts:

- complete bipartite graphs contain every fixed bipartite graph;
- affine geometries contain every fixed affine GF(q)-representable matroid.

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Bonin–Qin (2000): the binary case

$$\mathrm{ex}_2(\mathrm{AG}(m-1,2);n) < 2^{a_m n+1}, \qquad a_m = 1-2^{-(m-2)}, \qquad n \geq m \geq 3.$$

Every fixed binary affine matroid inherits an exponential saving.

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A natural question

For $q \geq 3$ and every fixed affine H, is there $\delta > 0$ such that

$$\mathrm{ex}_q(H;n) = O_{q,H}\left(q^{(1-\delta)n}\right)?$$

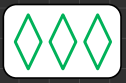
The game SET

Four features, three values each: $3^4 = 81$ different cards.

Shape	Color	Number	Shading
 Oval	 Red	 One	 Solid
 Squiggle	 Purple	 Two	 Striped
 Diamond	 Green	 Three	 Outlined

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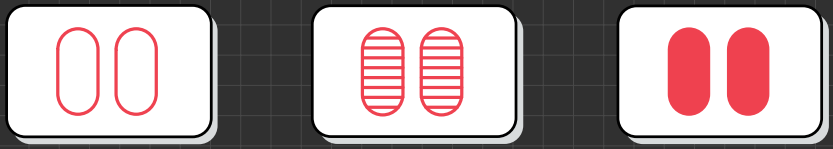
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A SET: three distinct cards, each feature all the same or all different.

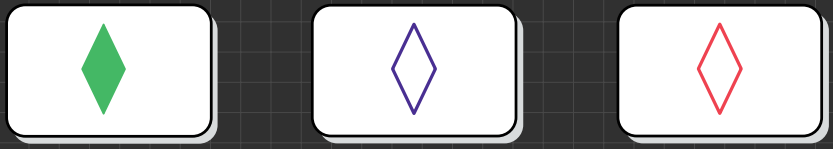
Is this a SET?

Check each feature separately.

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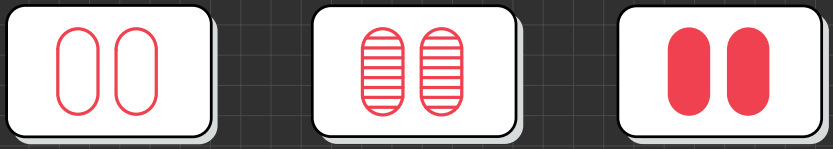
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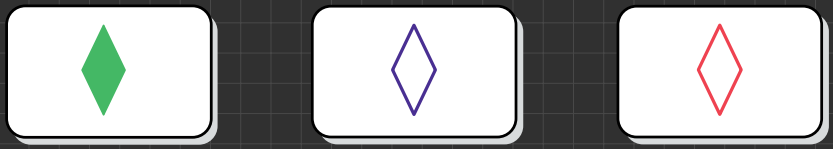
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SET: same color, shape and number. Shading all different.

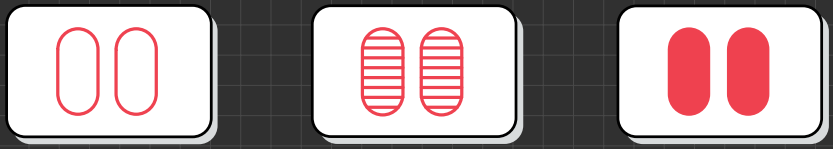
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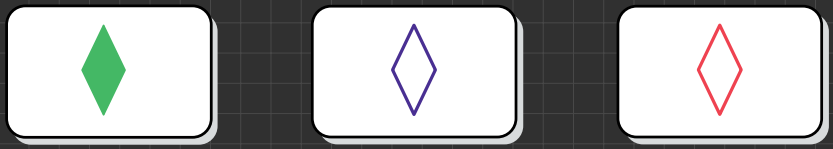
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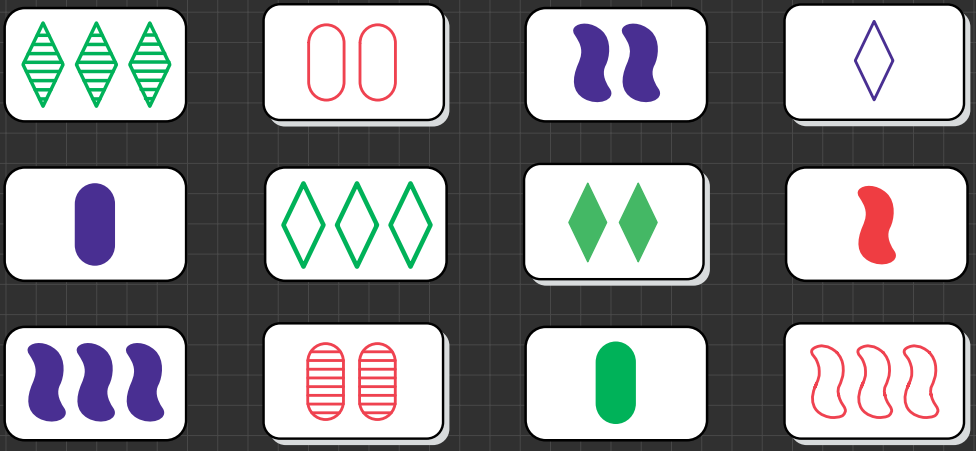
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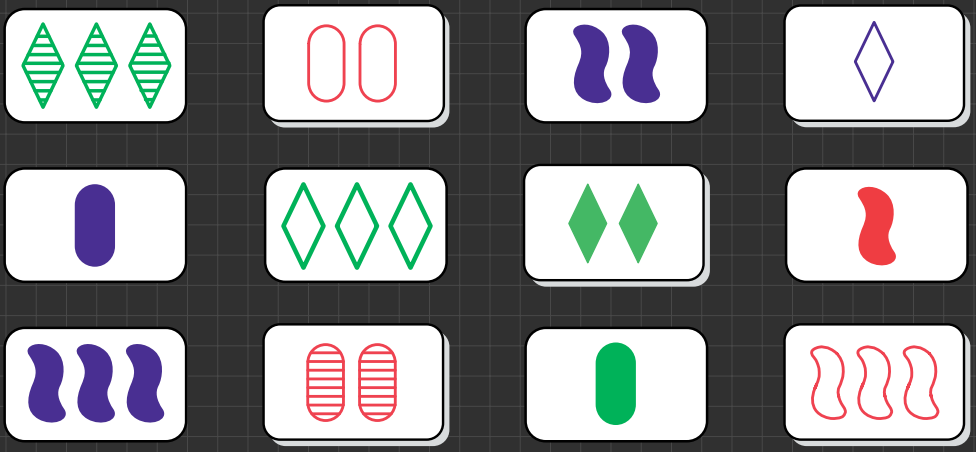


Not a SET: two outlined cards, one solid card.

Can you find a SET?

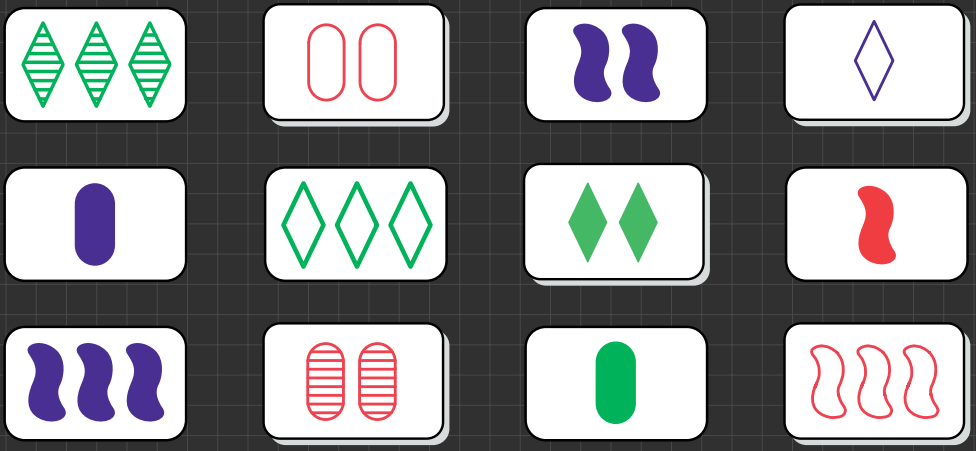


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How many cards can you collect without containing a SET?

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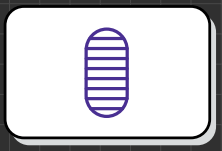


How many cards can you collect without containing a SET?

Answer: 20 — the largest SET-free collection (Pellegrino).

SET and affine geometry

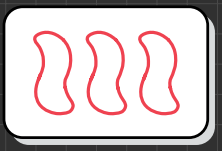
Encode each feature's values by $0, 1, 2$: the deck is $\mathbb{F}_3^4$.



$(0, 0, 0, 0)$



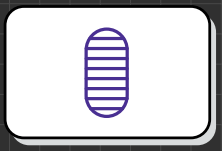
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$(2, 2, 2, 2)$

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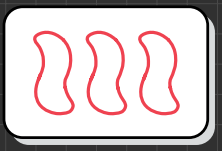
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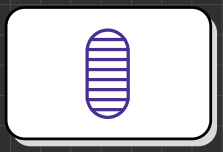
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Three distinct cards form a SET $\iff x + y + z = 0$.

Equivalently, they lie on an affine line.

SET and affine geometry

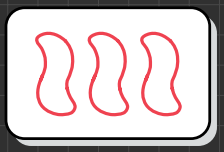
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The cap-set problem

How large can $A \subseteq \mathbb{F}_3^n$ be with no SET?

Cap sets and triangle-free matroids

$$\mathrm{cap}_3(n) = \max\{|A| : A \subseteq \mathbb{F}_3^n \text{ contains no affine line}\}.$$

$C_3(n) = \mathrm{ex}_3(M(K_3); n)$ uses matroids of rank n.

Choose projective representatives

Choose one nonzero vector per element.
A SET would give three dependent columns.

$$C_3(n) \leq \mathrm{cap}_3(n).$$

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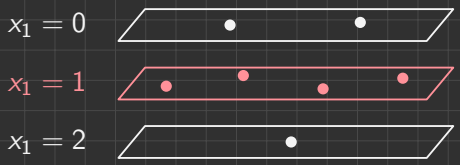
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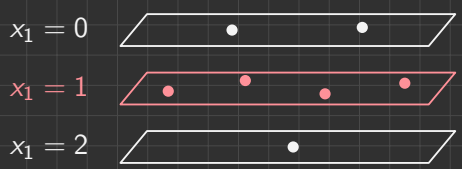
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Drop the fixed coordinate; lift to $(1, x)$ and extend to rank n.

$$\frac{1}{3}\mathrm{cap}_3(n) \le C_3(n) \le \mathrm{cap}_3(n), \qquad n \ge 1.$$

Two results over $\mathbb{F}_3$

Ellenberg–Gijswijt (2017): cap sets

$$\mathrm{ex}_3(M(K_3); n) = O\left(3^{(1-\delta)n}\right), \qquad \delta > 0.$$

Two results over $\mathbb{F}_3$

Ellenberg–Gijswijt (2017): cap sets

$$\mathrm{ex}_3(M(K_3); n) = O\left(3^{(1-\delta)n}\right), \qquad \delta > 0.$$

Fox–Pham (2019): higher-dimensional affine geometries

$$\mathrm{ex}_3(\mathrm{AG}(m-1, 3); n) = O_m\left(3^{(1-\delta_m)n}\right), \qquad \delta_m > 0$$

for each fixed $m \geq 2$.

Remark. Thanks to Bryce Frederickson for pointing me to these results.

Where we stand

Graphs	Matroids
Mantel; Turán	Bose–Burton
Erdős–Stone	Geelen–Nelson
Kővári–Sós–Turán	Bonin–Qin; Fox–Pham

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$M(K_{s,t})$ is affine over every finite field.

$$\mathrm{ex}_q(M(K_{s,t}); n) = ?$$

The matroidal Kővári–Sós–Turán bound

For every prime power q and fixed $2 \le s \le t$,

$$\boxed{\mathrm{ex}_q(M(K_{s,t}); n) = O_{q,s,t}\left(q^{(1-1/s)n}\right).}$$

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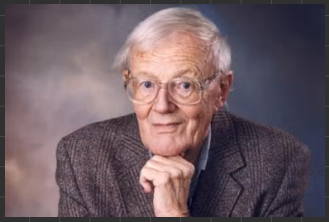
$M(K_{s,t})$ is affine over every $\mathrm{GF}(q)$.

Compare, for $q = 2$, Bonin–Qin:

$$\mathrm{ex}_2(M(K_{s,t}); n) = O\left(2^{(1-2^{-(s+t-3)})n}\right).$$

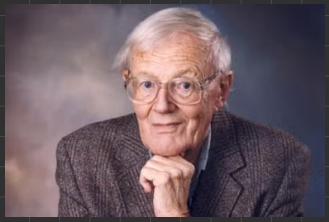
The saving improves from $2^{-(s+t-3)}$ to $1/s$.

Proof sketch: reproduce the s-star count.



W. T. Tutte

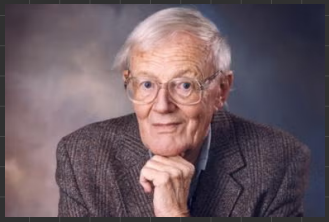
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J. G. Oxley

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How can we translate the proof?

Graph proof	Matroid setting
Edges	Matroid elements
Vertices	???
Common neighbors	???

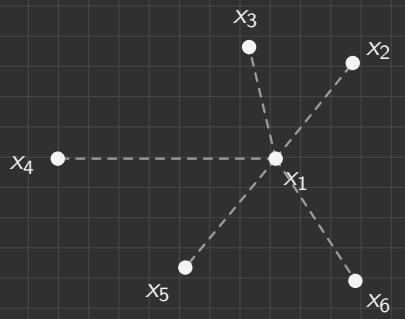
How can we translate the proof?

Graph proof	Matroid setting
Edges	Matroid elements
Vertices	An affinely independent s-tuple X of vectors
Common neighbors	Translates y with $X + y \subseteq A$

A set of s vertices

Fix $\mathbb{F}_q$;
$E(M) = A \subseteq \mathbb{F}_q^n$, a set of vectors.

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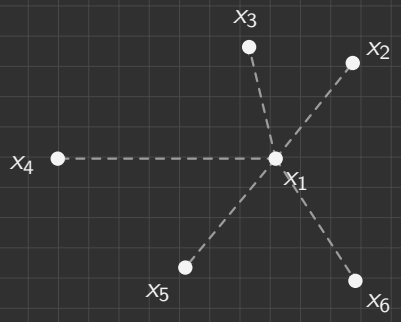
A "set of s vertices" is an ordered tuple

$$X = (x_1, x_2, \ldots, x_s) \in (\mathbb{F}_q^n)^s$$

of affinely independent vectors, i.e.

$x_2 - x_1, \ldots, x_s - x_1$ are linearly independent.

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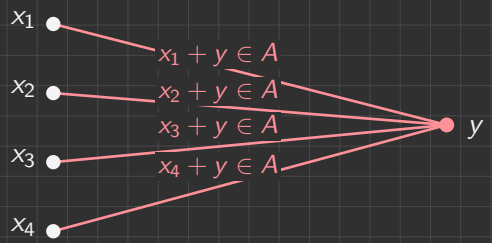
$x_2 - x_1, \ldots, x_s - x_1$ are linearly independent.

The vectors x_i need not lie in A.

Common neighbors

$y \in \mathbb{F}_q^n$ is a common neighbor of $X = (x_1, \ldots, x_s)$ if $x_i + y \in A$ for every i.

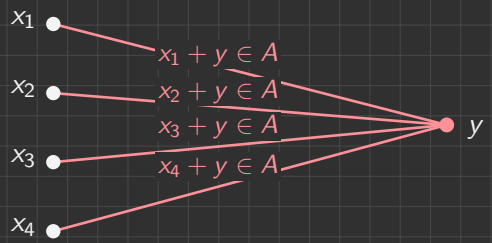
Write $N(X)$ for the set of common neighbors.



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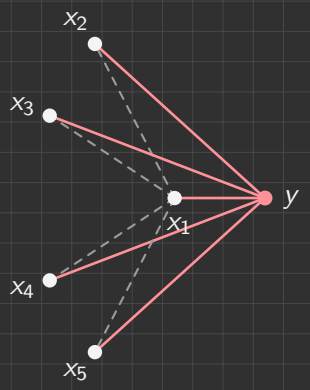


Not every common neighbor gives an $M(K_{1,s})$.

When does a common neighbor give $M(K_{1,s})$?

Put $v_i = x_i - x_1$ and

$$W = \operatorname{span}\{v_2, v_3, \ldots, v_s\}, \qquad \dim W = s - 1.$$



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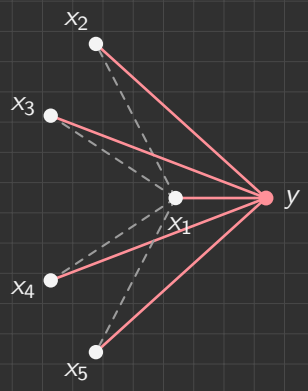
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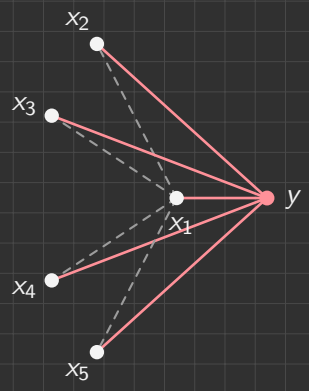
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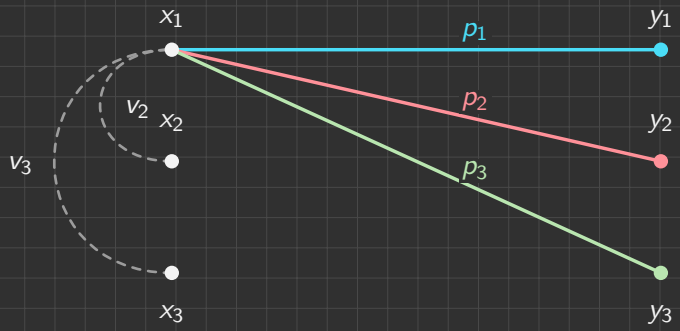
exactly when $x_1 + y \notin W$.



An independent spanning tree

To combine the stars, use an independent tree.

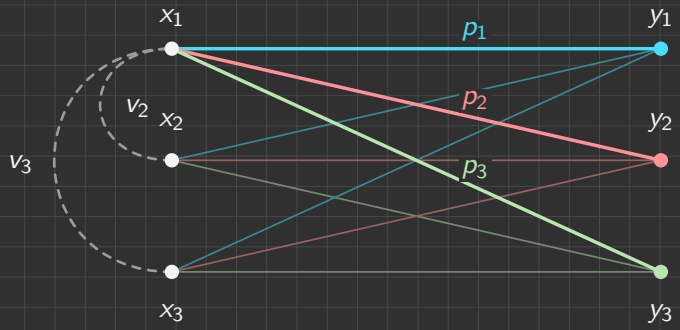
Let $v_1 = 0$ and let v_2, v_3, p_1, p_2, p_3 be independent.



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$v_i + p_j \in A$ for $1 \leq i, j \leq 3 \Rightarrow$ an $M(K_{3,3})$-restriction.

Many neighbors $\Rightarrow$ a forbidden $M(K_{s,t})$

If there exist $y_1, \ldots, y_t \in N(X)$ such that

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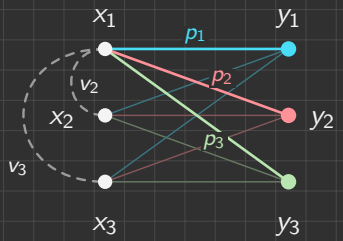
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$$x_i + y_j = v_i + p_j \in A$$



	y_1	y_2	y_3
x_1	p_1	p_2	p_3
x_2	$v_2 + p_1$	$v_2 + p_2$	$v_2 + p_3$
x_3	$v_3 + p_1$	$v_3 + p_2$	$v_3 + p_3$

A constant bound on common neighbors

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$$\boxed{|N(X)| \leq |U| \leq q^{s-1} \cdot q^{t-1} = q^{s+t-2}.}$$

Double count the pairs (X, y) with $y \in N(X)$

Upper bound: count by the leaves X

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Translation preserves affine independence, so

$$\#(X, y) = \big|\{\text{affinely independent } s\text{-tuples in } A^s\}\big| \cdot q^n.$$

For fixed q, s and sufficiently large $|A|$, there are $\Theta_{q,s}(|A|^s)$ such tuples.

Compare the two counts

By the leaves X $\#(X, y) \leq q^{ns} \cdot q^{s+t-2}$

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$$|E(M)| = |A| = O_{q,s,t}\left(q^{(1-1/s)n}\right)$$

Binary Sidon sets

Binary Sidon set $S \subseteq \mathbb{F}_2^n$: no distinct $a, b, c, d \in S$ with $a + b = c + d$, i.e. no $M(C_4) = M(K_{2,2})$ (for $0 \notin S$).

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Sidon sets with $|S| = \Omega(2^{n/2})$ exist, so

$$\boxed{\mathrm{ex}_2(M(K_{2,t}); n) = \Theta_t(2^{n/2})} \qquad (t \geq 2).$$

The exponent $1 - 1/s$ is sharp for $s = 2$, $q = 2$.

Beyond complete bipartite graphs

Let H be bipartite with a cycle and degree at most s on one side.

Graphs | Matroids

Alon–Krivelevich–Sudakov (2003)

$$\mathrm{ex}(H; n) = O_H(n^{2-1/s})$$

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$\mathrm{ex}(C_{2k};n) = O_k(n^{1+1/k})$	$\longrightarrow$	$\mathrm{ex}_q(M(C_{2k});n) = O_{q,k}(q^{n/k})$

For $k \geq 2$ and $q = 2$ this is sharp: $\mathrm{ex}_2(M(C_{2k});n) = \Theta_k(2^{n/k})$.

What would be good to know?

Forbidding complete p-partite graphs, for some $p \leq q$

$M(K_{a_1,\ldots,a_p})$ is affine over GF(q), so by Geelen–Nelson

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Triangles, $K_3 = K_{1,1,1}$: Ellenberg–Gijswijt (2017), $q \geq 3$

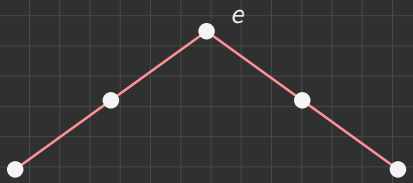
$$C_q(n) = \mathrm{ex}_q(M(K_3); n) = O\left(q^{(1-\delta_q)n}\right), \qquad \delta_q > 0.$$

Triangles and $K_{1,1,t}$

In $\mathbb{F}_q$, start with $K_{1,1,1}$ and $K_{1,1,2}$.



$M(K_{1,1,1}) = M(K_3)$



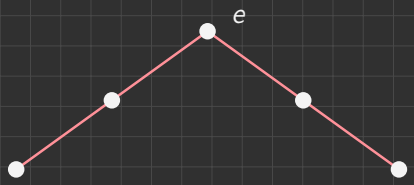
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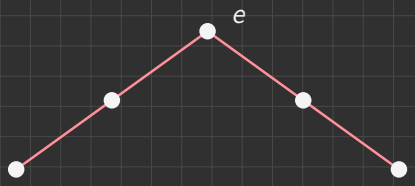
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$$C_q(n) \leq \mathrm{ex}_q(M(K_{1,1,2}); n) \leq \frac{q+1}{2} C_q(n).$$

Recreational problems

For fixed q and $t \geq 2$, can we keep the same cap-set growth?

$$\mathrm{ex}_q(M(K_{1,1,t}); n) = O_{q,t}(C_q(n)) \ ?$$

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Let the singleton parts grow: $K_{1,1,t} \longrightarrow K_{a,b,c}$, $1 \leq a \leq b \leq c$.

A consequence of Fox–Pham (2019): for some $\delta = \delta(a, b, c) > 0$,

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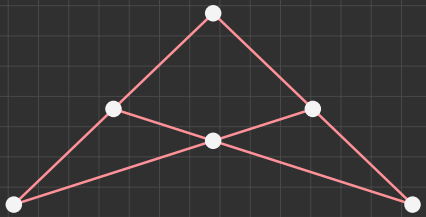
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How does the best δ depend on a, b, c?

Can the saving be independent of c?

First targets over $\mathbb{F}_4$

Both targets are affine over GF(4).



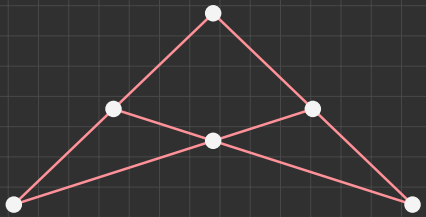
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For either target N, can we force exponential loss?

$$\mathrm{ex}_4(N; n) = O_N(4^{(1-\delta)n}), \quad \delta > 0 \ ?$$

Rational exponents

Erdős–Simonovits: the rational exponents conjecture (now Theorem)

For every rational $r \in [1, 2]$ there is a graph H with

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The dictionary so far

KST	$n^{2-1/s}$	$\longleftrightarrow$	$q^{(1-1/s)n}$
Bondy–Simonovits	$n^{1+1/k}$	$\longleftrightarrow$	$q^{n/k}$
In general	$n^{1+\beta}$	$\longleftrightarrow$	$q^{\beta n}$

Rational exponents for matroids

Conjecture

For every prime power q and every rational $\beta \in (0,1)$,
there is an affine simple GF(q)-representable matroid H with

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A first step

A matroid Bukh–Conlon: realize every rational β by a finite family.

Summary

- Turán problems for GF(q)-representable matroids mirror graphs: Bose–Burton $\leftrightarrow$ Turán, Geelen–Nelson $\leftrightarrow$ Erdős–Stone.
- Affine targets play the role of bipartite graphs; Geelen–Nelson gives only $o(q^n)$.
- Matroidal KST: $\mathrm{ex}_q(M(K_{s,t}); n) = O_{q,s,t}(q^{(1-1/s)n})$, sharp for $s = q = 2$ (Sidon sets).
- Proof: vertices $\to$ affinely independent tuples; common neighbors $\to$ translates.
- Open: exponential savings for affine targets, e.g. $M(K_4)$ and $U_{2,4}$ over $\mathbb{F}_4$.
- Conjecture: every rational $\beta \in (0, 1)$ occurs as $\mathrm{ex}_q(H; n) = \Theta(q^{\beta n})$, H affine.

https://playset.netlify.app

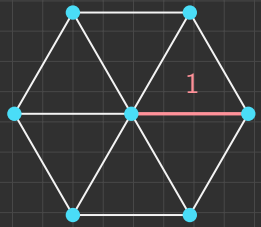
Backup

An application of KST: the Erdős unit-distance problem

The Erdős unit-distance problem

How many unit-distance pairs can n points in $\mathbb{R}^2$ determine?

Hexagon and center

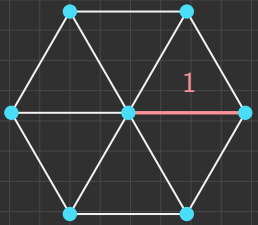


7 points, 12 unit pairs

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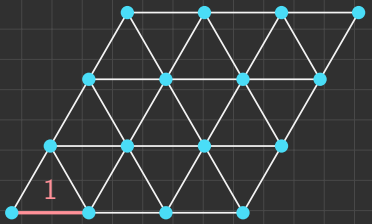
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Triangular lattice

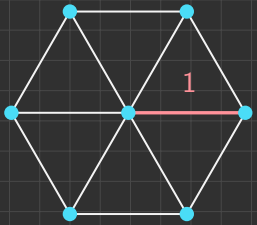


16 points, 33 unit pairs

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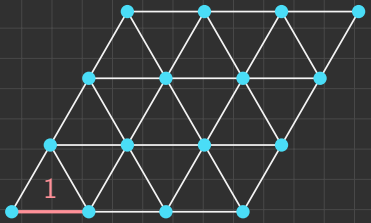
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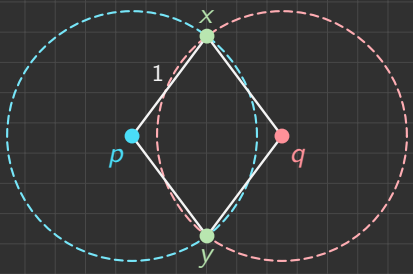
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Write $u(n)$ for the maximum number of unit-distance pairs.

$$u(n) = \max_{P \subseteq \mathbb{R}^2,\ |P|=n} \#\{\{x, y\} \subseteq P : \|x - y\| = 1\}.$$

Common neighbors in the unit-distance graph

Unit-distance graph $G(P)$: vertices P; edges xy with $\|x - y\| = 1$.

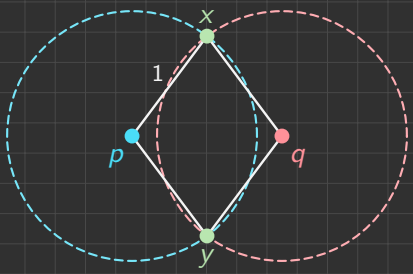


Fix distinct points $p, q \in P$.

A common neighbor must lie on both unit circles.

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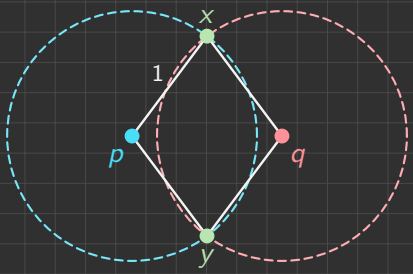
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$$|N(p) \cap N(q)| \leq 2.$$

Two distinct circles have at most two intersection points.

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$G(P)$ is $K_{2,3}$-free.

By KST: $u(n) \leq \mathrm{ex}(K_{2,3}; n) = O(n^{3/2})$.