

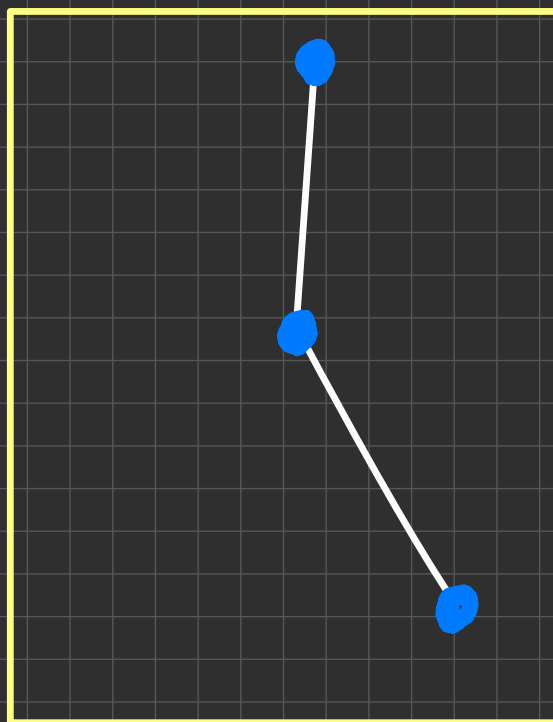
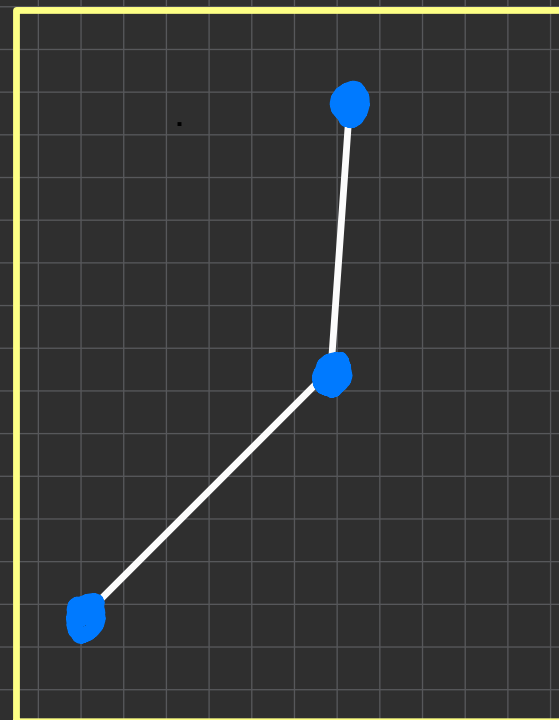
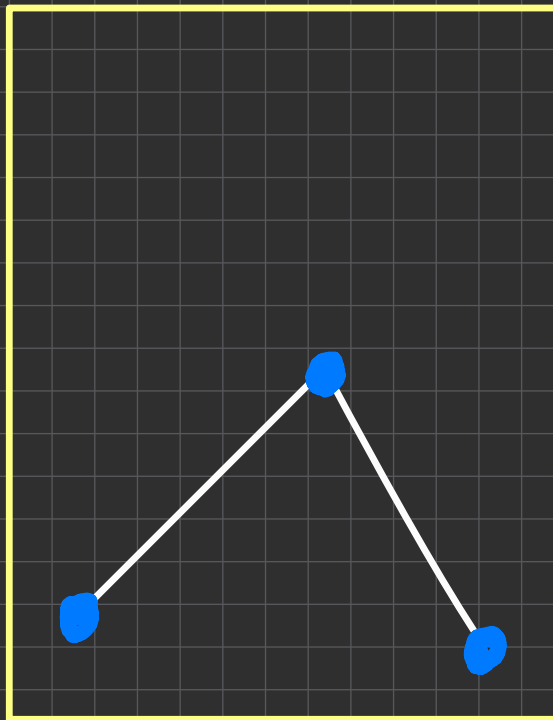
Reconstruction with Digitally Convex Sets

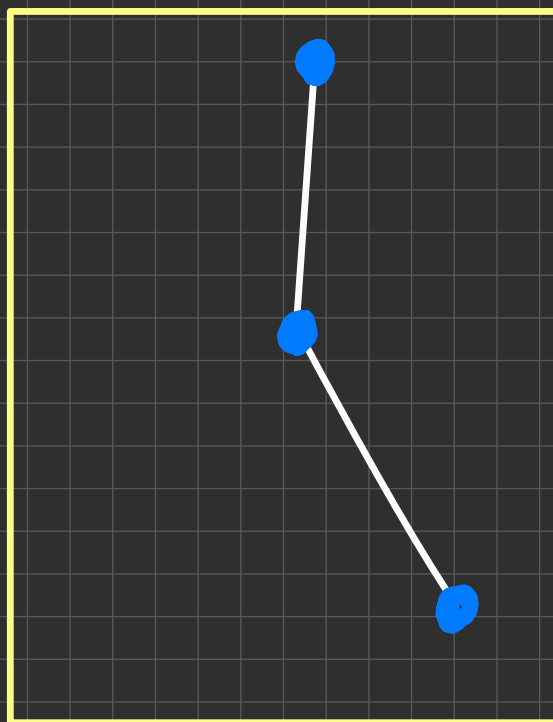
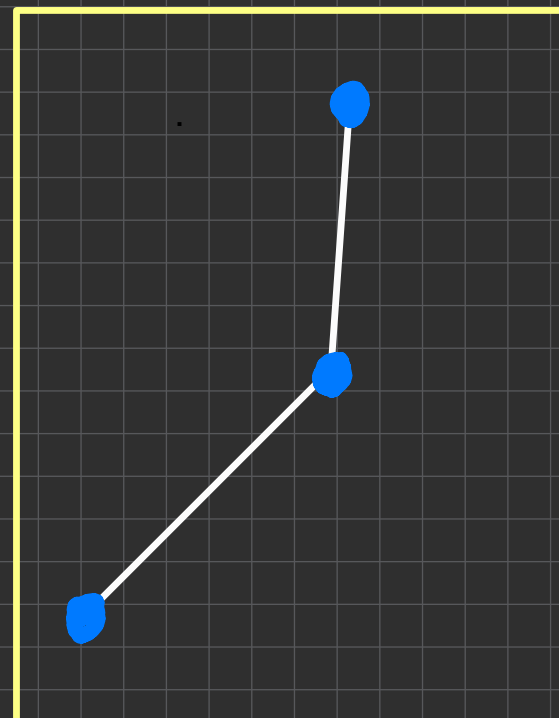
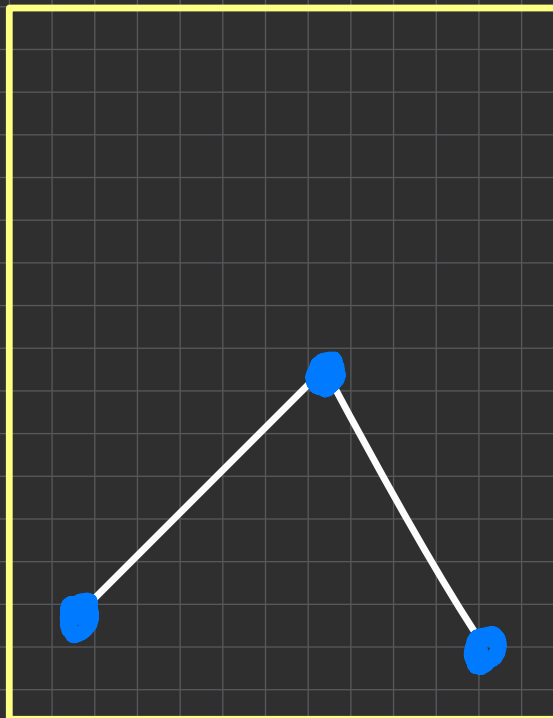
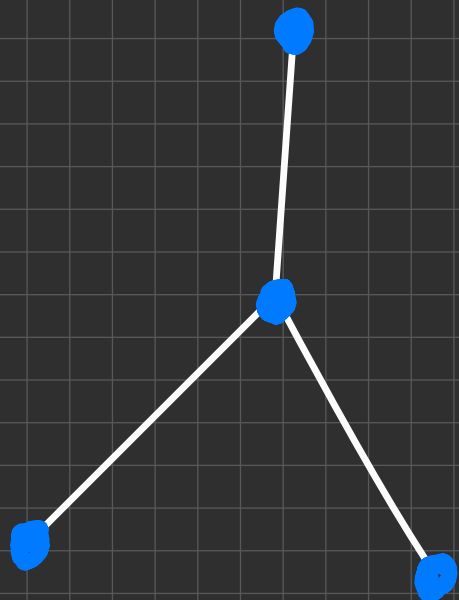
LSU Combinatorics Seminar

Yiwei Ge (LSU)

joint with a group from GRWC

What is this graph?



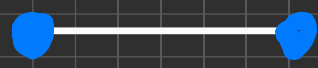
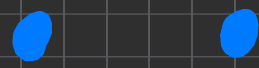


the reconstruction conjecture (Kelly, 1942; Ulam, 1960)

every graph of order at least 3 can be determined uniquely, up to isomorphism, from the collection of its 1-vertex deleted subgraphs.

the reconstruction conjecture (Kelly, 1942; Ulam, 1960)

every graph of order at least 3 can be determined uniquely, up to isomorphism, from the collection of its 1-vertex deleted subgraphs.

 and  have the same deck of cards

Reconstructable classes:

- Regular graphs (Harary, 1974)
- Trees (Harary, 1974)
- Disconnected graphs (Harary, 1974)
- Unit interval graphs (von Rimscha, 1983)
- Planar triangulations (Fiorini and Lauri, 1981)
- Outerplanar graphs (Giles, 1974)

...

the ^{edge} reconstruction conjecture

the ^{edge} reconstruction conjecture (Harary, 1964)

Any two graphs with at least four edges
and having the same edge-decks are isomorphic.

the ^{edge} reconstruction conjecture (Harary, 1964)

Any two graphs with at least four edges and having the same edge-decks are isomorphic.

for matroid analogues, see Oxley Sec 15.4

What is RECONSTRUCTION?

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partial information $\Rightarrow$ graph

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partial information $\Rightarrow$ graph

- Kelly, Ulam, Harary unlabelled graph

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partial information $\Rightarrow$ graph

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difficult; hardly touched

$\Rightarrow$ still conjectures

What is RECONSTRUCTION?

partial information $\Rightarrow$ graph

- Kelly, Ulam, Harary unlabelled graph
for labelled graph?

What is RECONSTRUCTION?

partial information $\Rightarrow$ graph

- Kelly, Ulam, Harary unlabelled graph
for labelled graph?

Need to restrict partial information

$$N[v] = N_G(v) \cup \{v\}$$

$$N[S] = \bigcup_{v \in S} N[v]$$

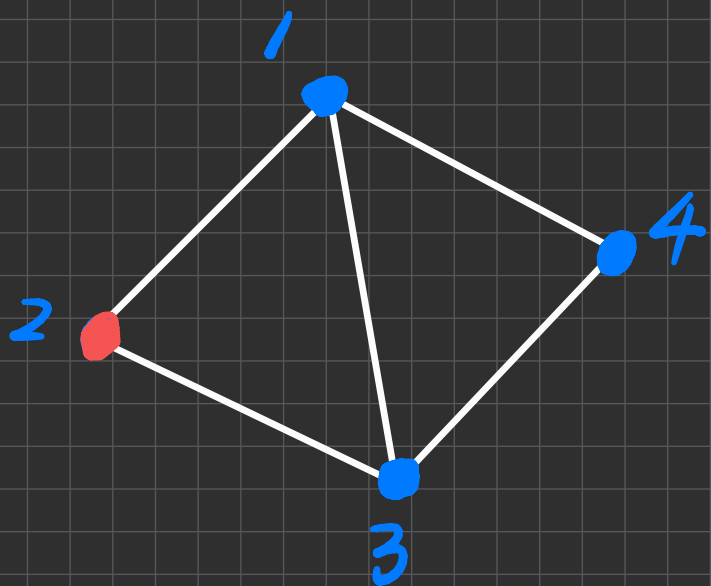
A set $S \subseteq V(G)$ is a digitally convex set
if for every vertex $v \notin S$, $N[v] \setminus N[S] \neq \emptyset$.

$\emptyset$ and $V(G)$ are always digitally convex

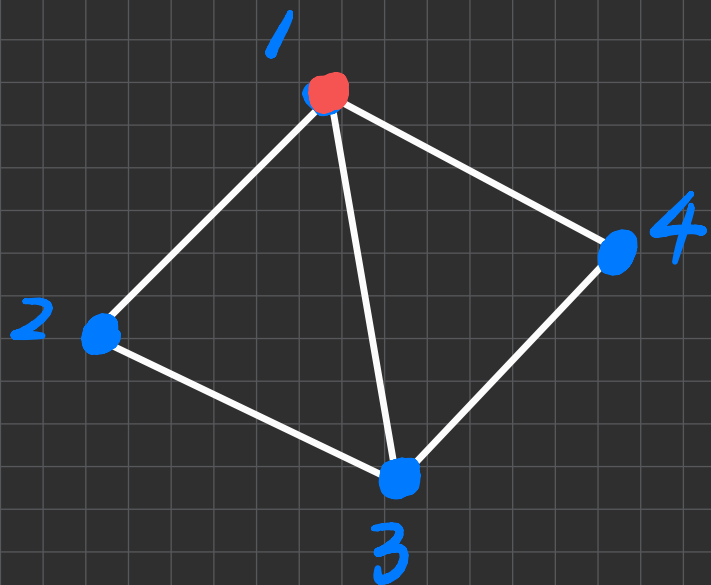
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$$x \in N[v] \setminus N[S]$$

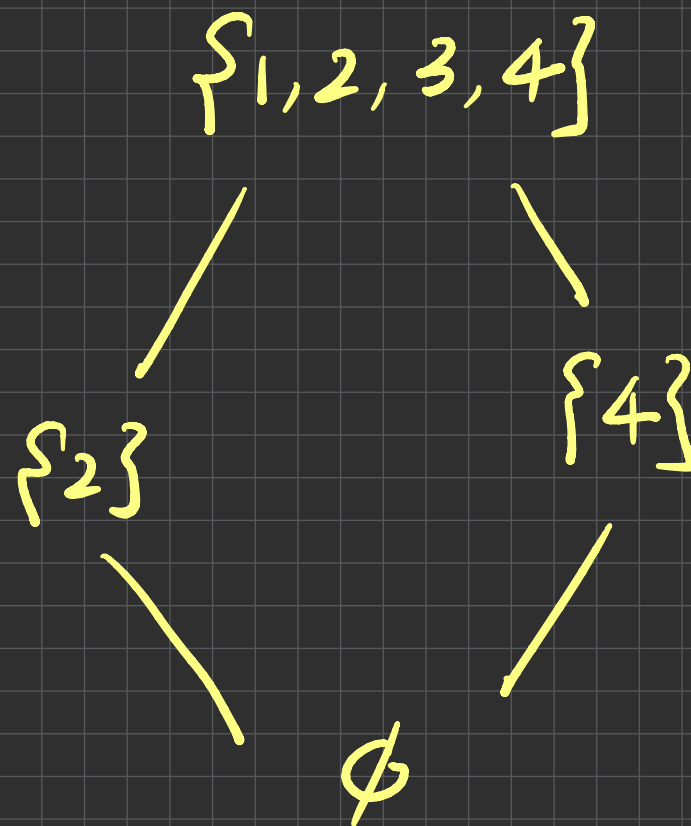
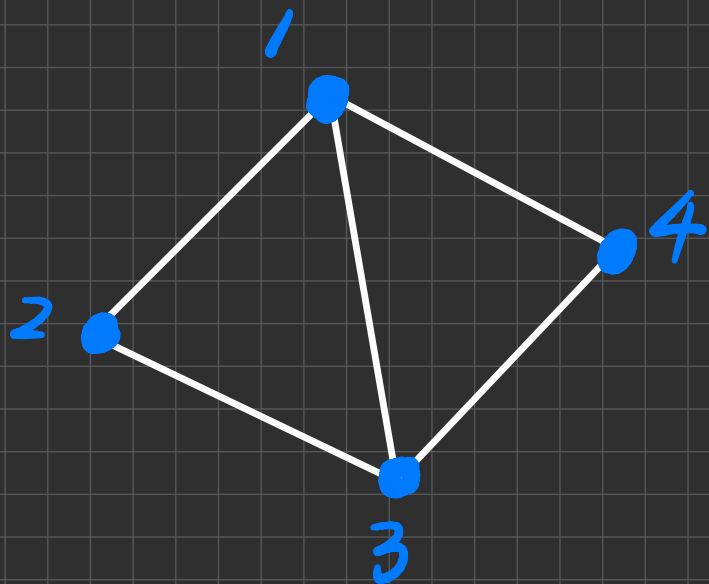
“private neighborhood”



$\{2\}$ is digitally convex



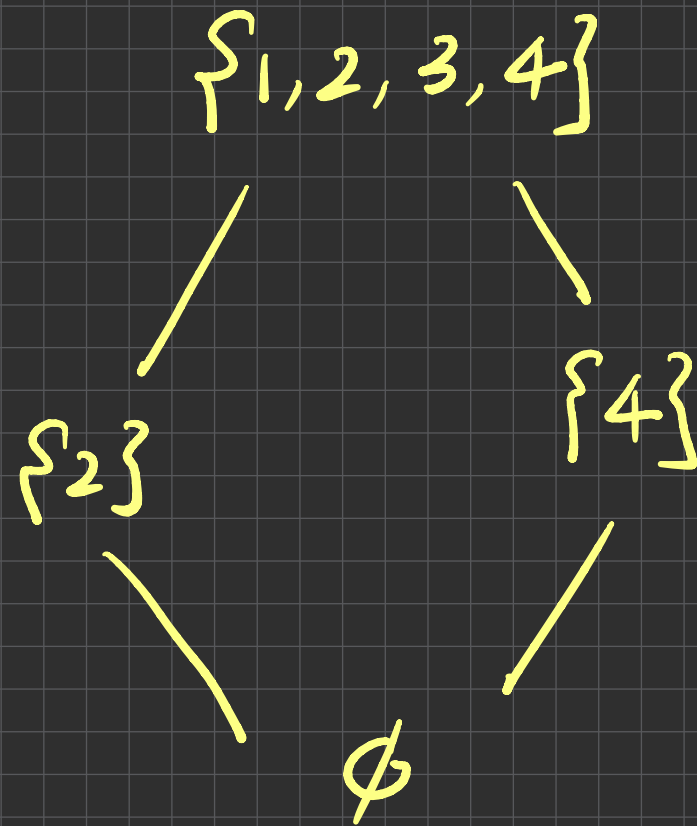
$\{1\}$ is not



$\mathcal{P}(G) :=$ collection of d.c.s
of G

Note: $\mathcal{P}(G)$ is closed under taking intersection

What is the graph?



Can we reconstruct any graph G from
its collection of digitally convex sets $\mathcal{L}(G)$?

Can we reconstruct any graph G from its collection of digitally convex sets $\mathcal{L}(G)$?

- Yes, if G is a tree.
- Yes, if the girth of G is at least 7.

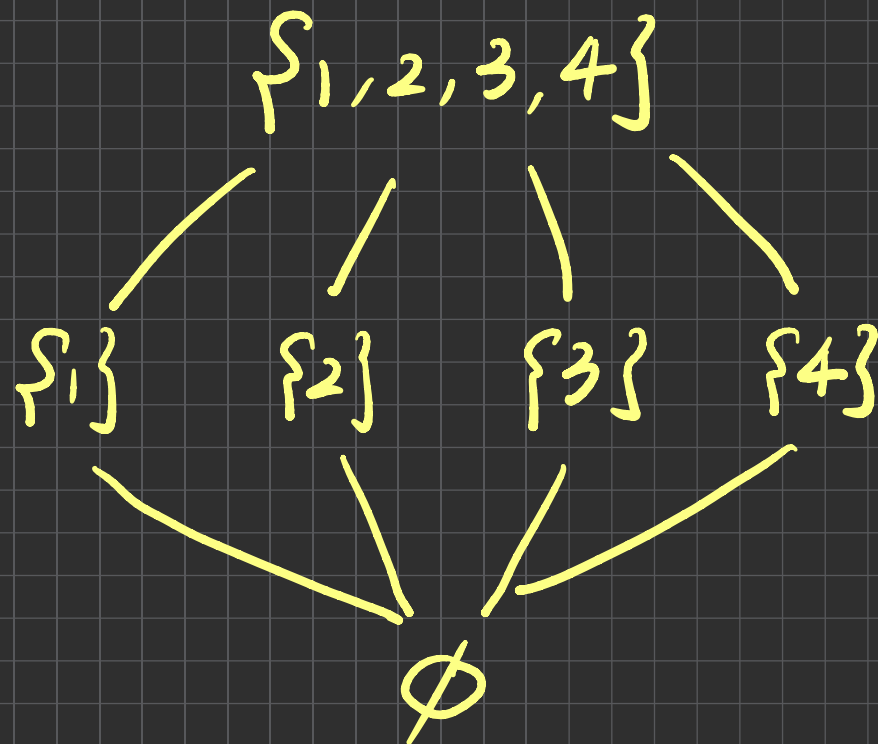
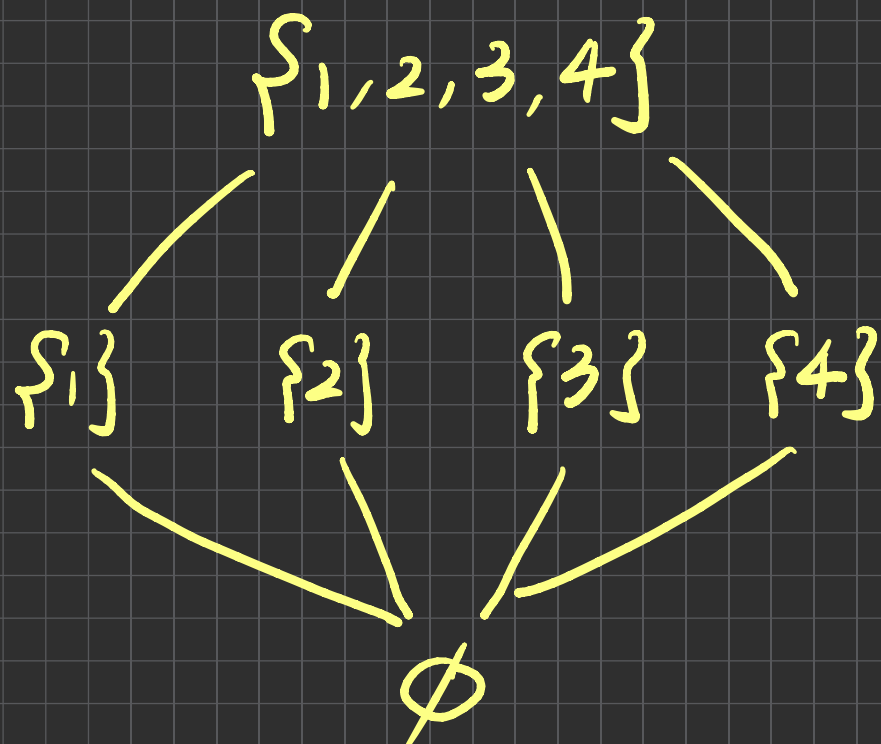
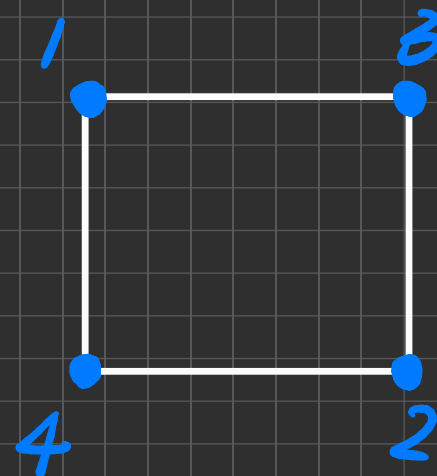
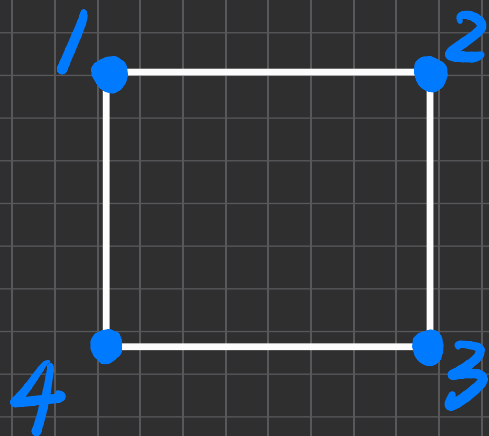
(Lafrairie, Oellermann, Pressey 2014)

Can we reconstruct any graph G from
its collection of digitally convex sets $\mathcal{L}(G)$?

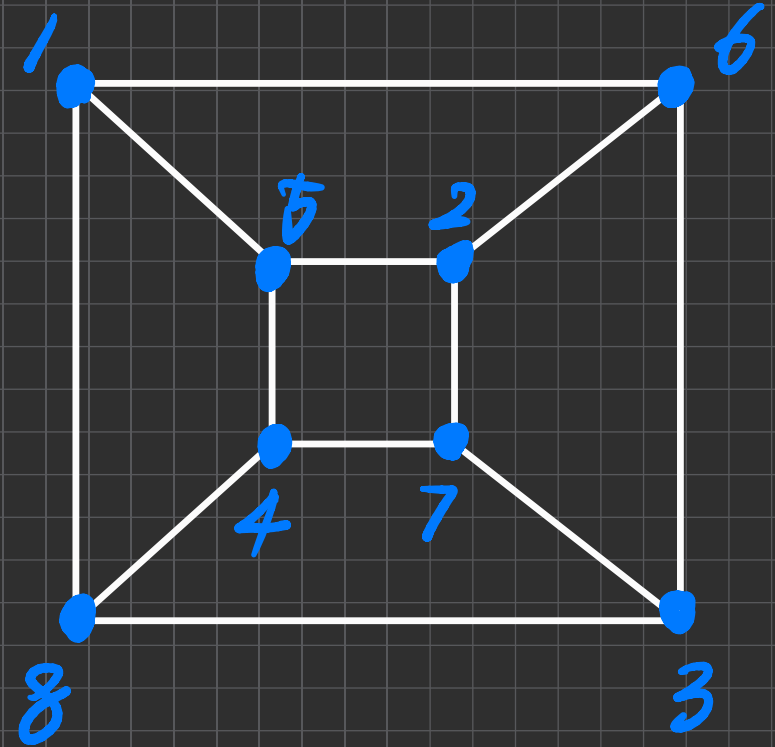
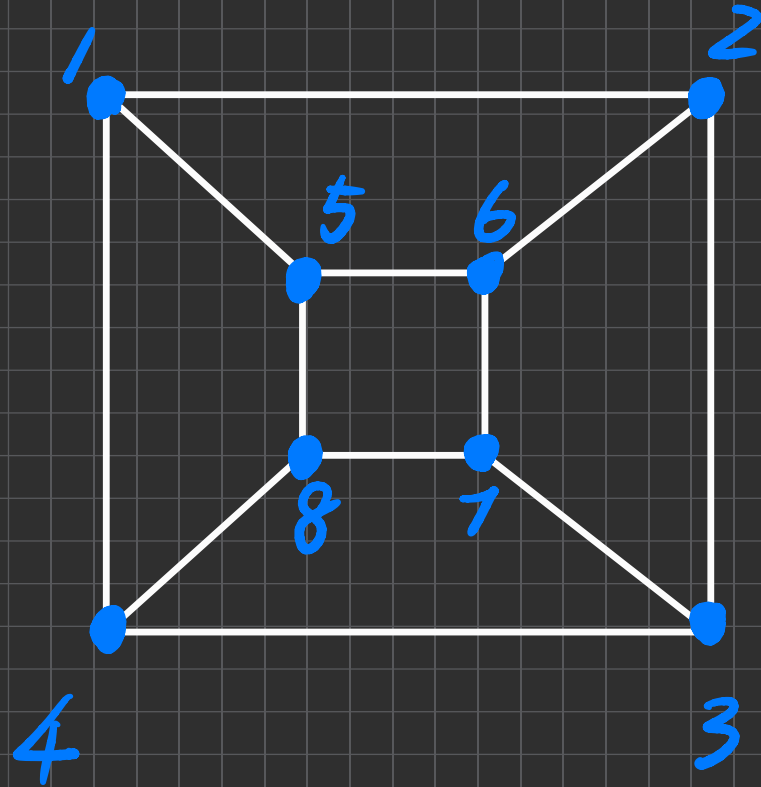
In general,

NO :c

Counterexample



Move counterexamples



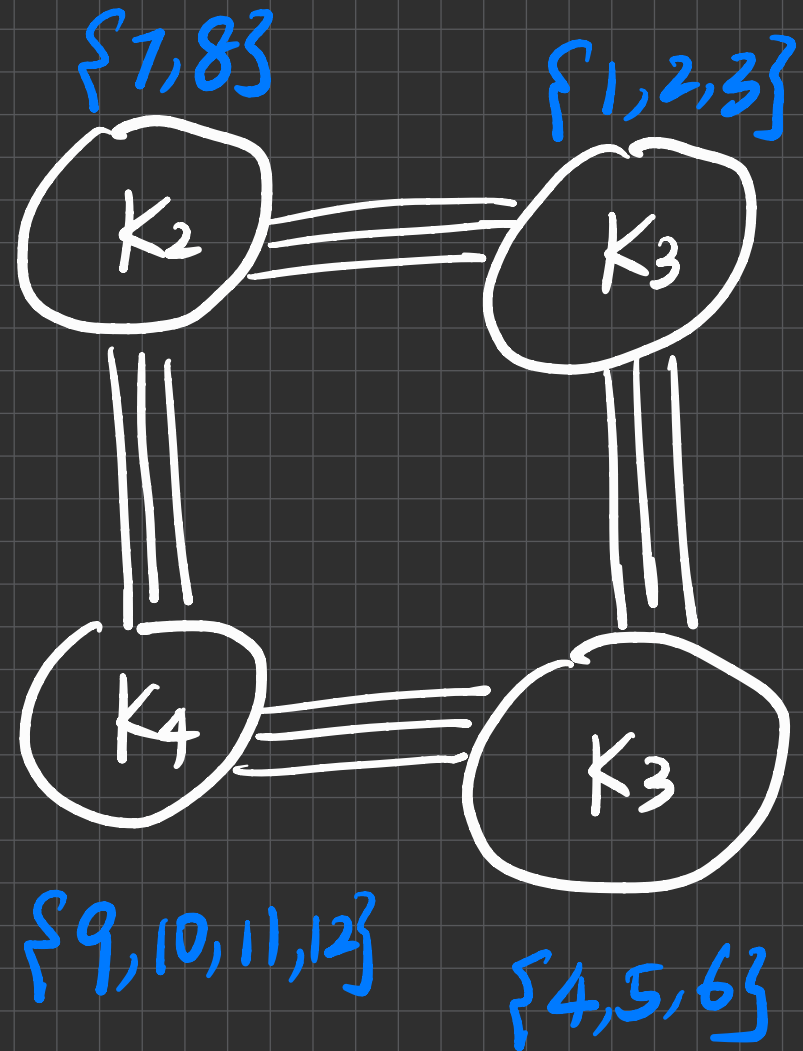
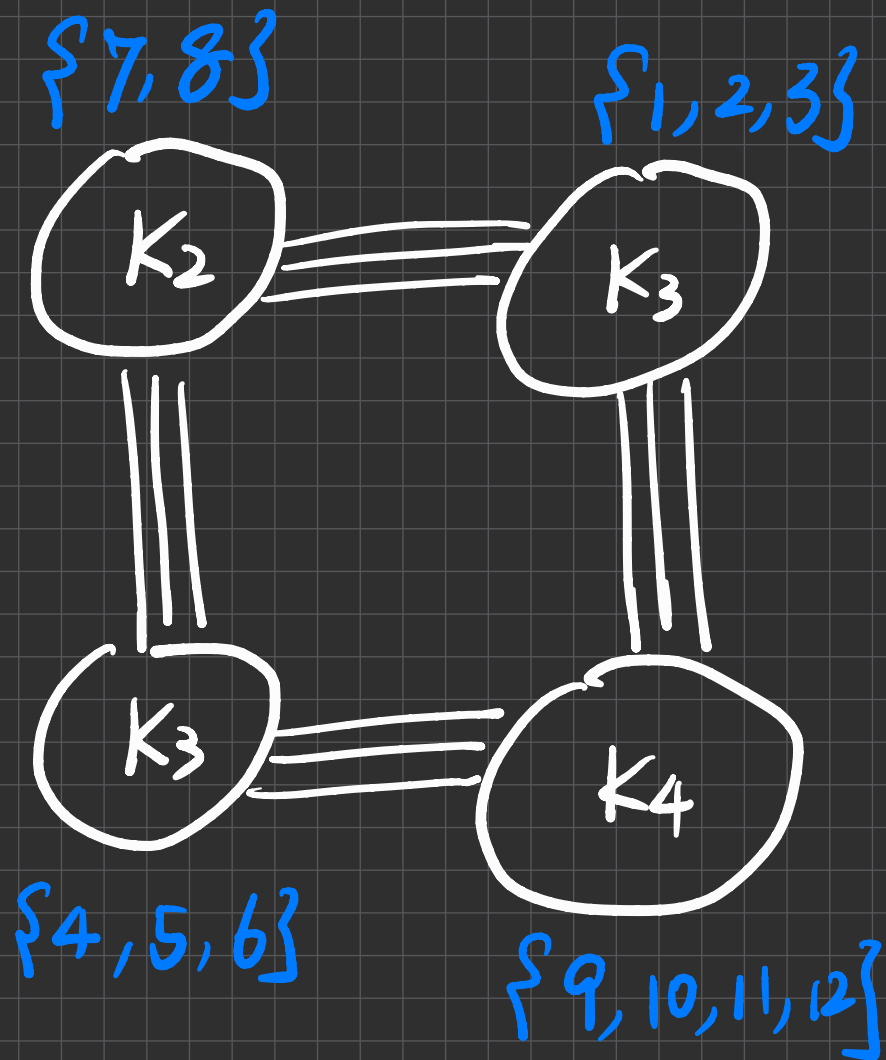
higher dimension hypercubes ...

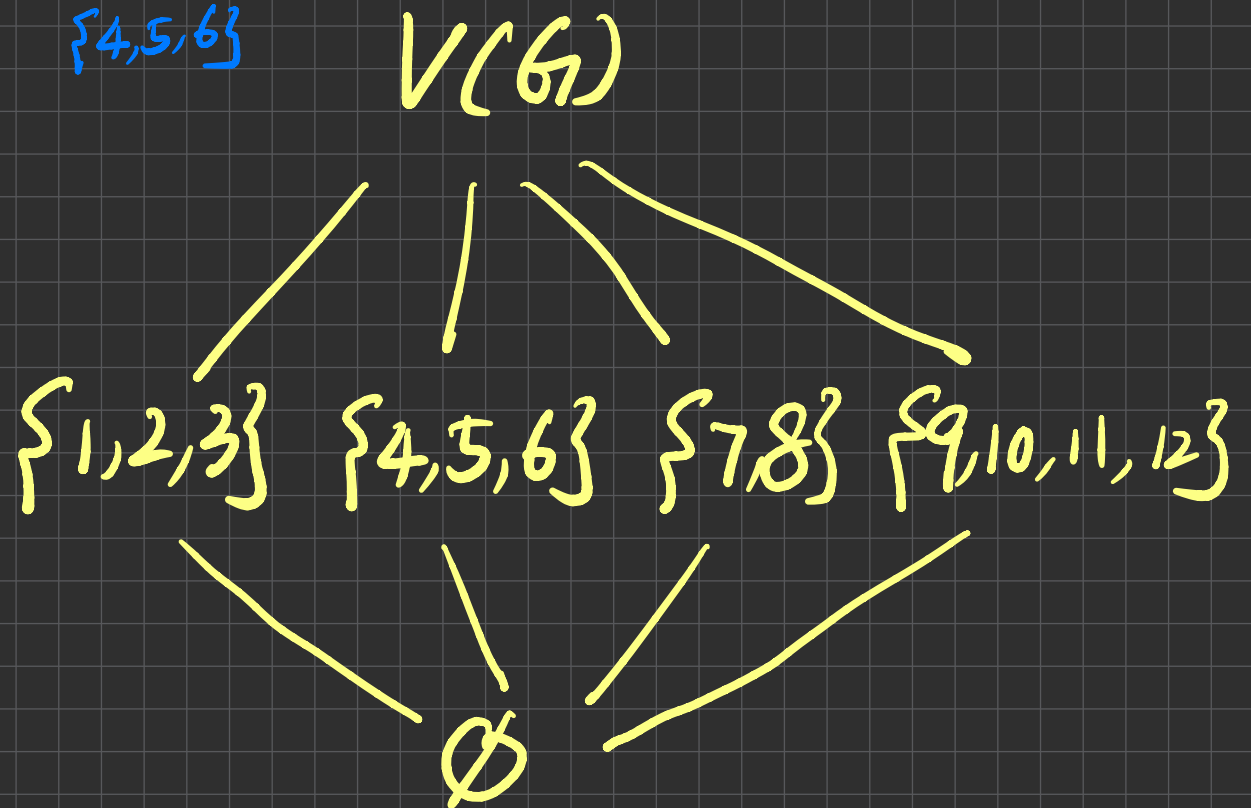
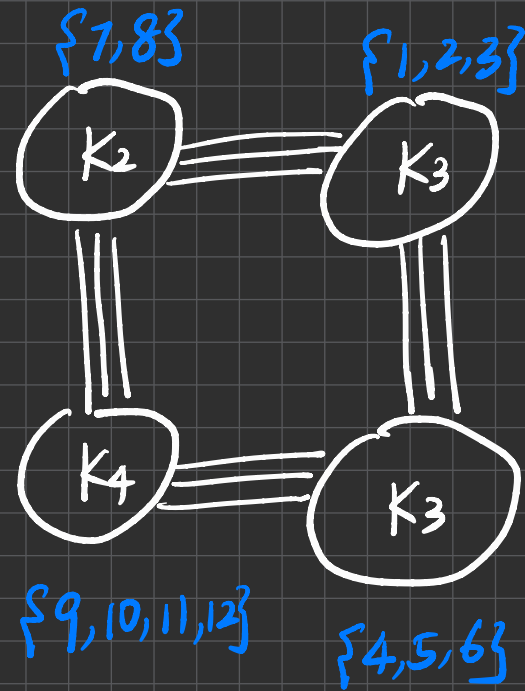
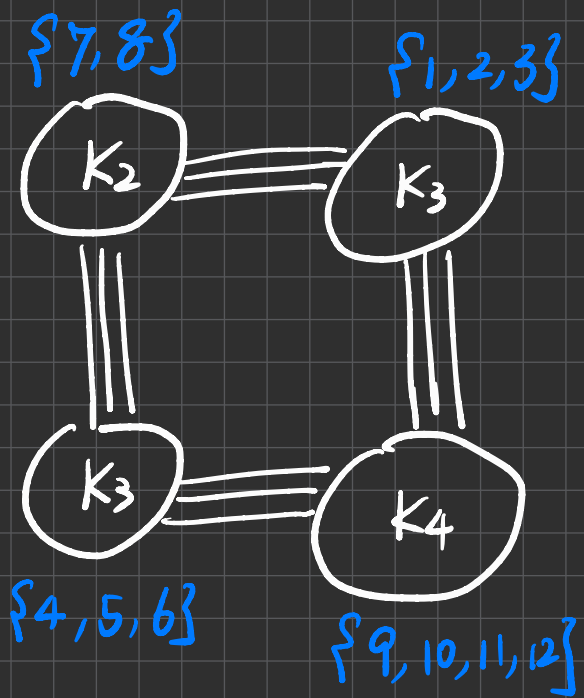
Reconstructing an isomorphic graph
of G from $\mathcal{L}(G)$?

Reconstructing an isomorphic graph
of G from $\mathcal{L}(G)$?

NO :c

Counterexample





Our main result:

Let G be an induced- C_4 -free graph and $\mathcal{P}(G)$ be its collection of digitally convex sets. Then, G is reconstructable from $\mathcal{P}(G)$.

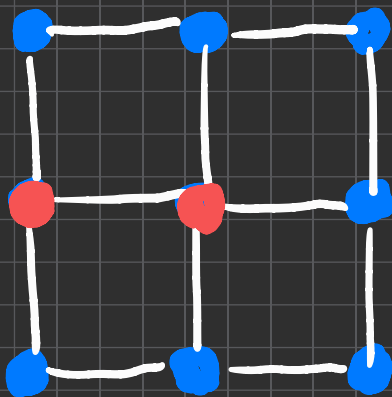
What is a digitally convex set?

$$S \in \mathcal{C}(G) \iff \forall v \notin S, \exists x \in N[v] - N[S]$$

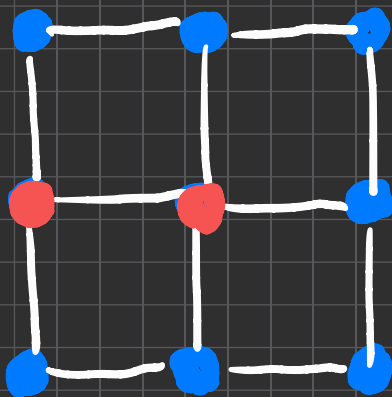
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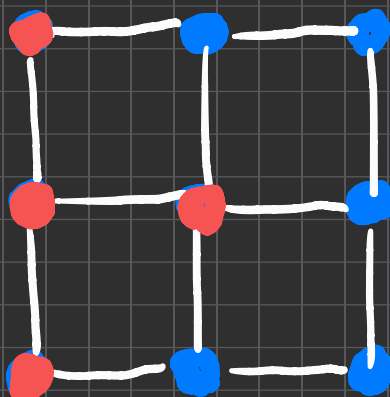
What is $\bar{S}$ ($:= V(G) - S$)?



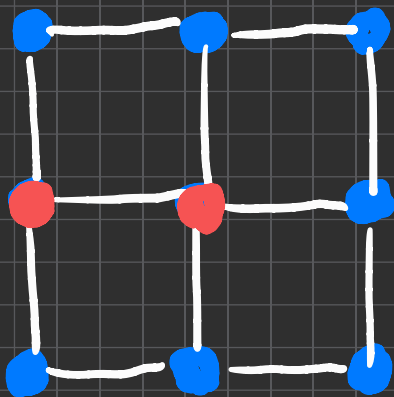
A



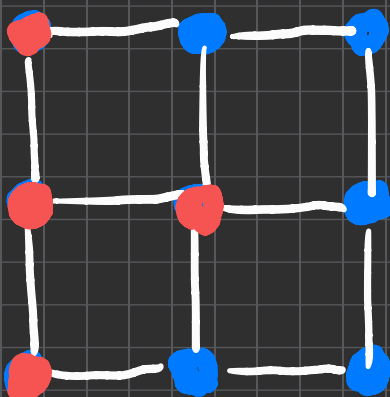
A



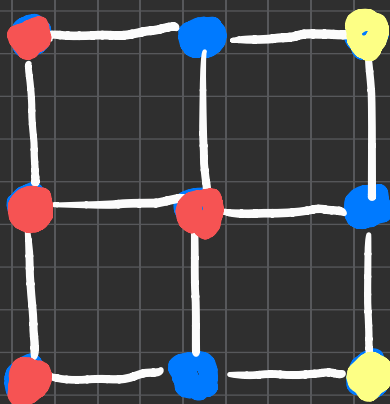
Conv(A)



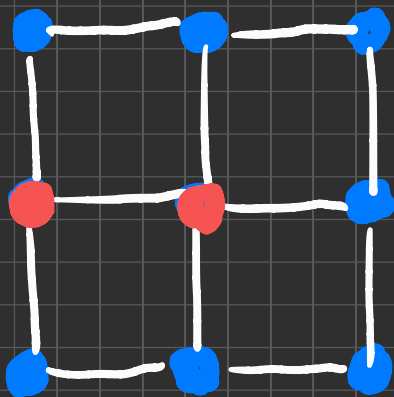
A



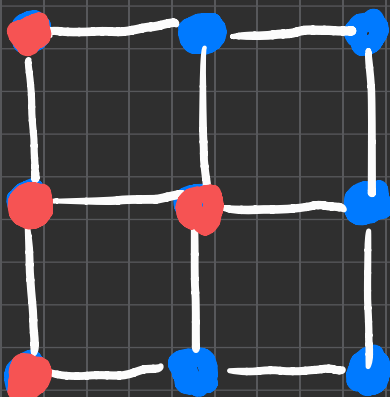
$\text{Conv}(A)$



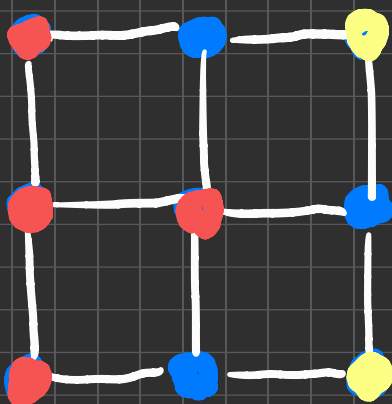
X



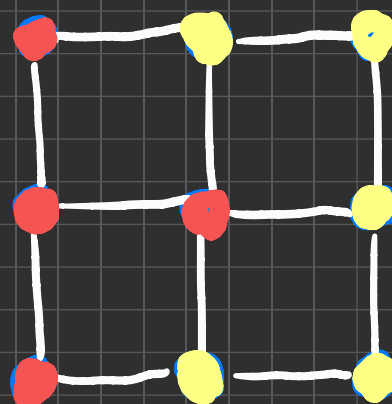
A



Conv(A)



X



Conv(A) = N[x]

What is a digitally convex set?

$$S \in \mathcal{C}(G) \iff \forall v \notin S, \exists x \in N[v] - N[S]$$

What is $\bar{S}$ ($:= V(G) - S$)?

Any vertex $x \in V(G) - N[S]$ is a private nbhd

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What is $\bar{S}$ ($:= V(G) - S$)?

Any vertex $x \in V(G) - N[S]$ is a private nbhd

$$\forall S \in \mathcal{C}(G), \exists X \subseteq V(G) \text{ s.t. } \bar{S} = N[X].$$

collection of closed neighborhood $\mathcal{N}(G)$

$$\mathcal{N}(G) = \{N[A] : A \subseteq V(G)\}$$

collection of closed neighborhood $\mathcal{N}(G)$

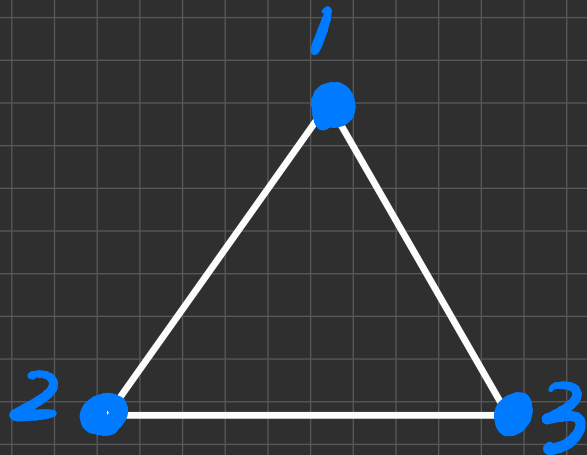
$$\mathcal{N}(G) = \{N[A] : A \subseteq V(G)\}$$

Cryptomorphism

Knowing $\mathcal{L}(G)$ is same as knowing $\mathcal{N}(G)$

$$\therefore \mathcal{L}(G) = \overline{\mathcal{N}(G)}$$

$$\mathcal{N}^*(G) := \{N[v] : v \in V(G)\} \quad \text{multi-set}$$



$$\mathcal{N}^*(G) = \{ \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\} \}$$

Lemma 1 .

Let G be an induced- C_4 -free graph,
then G is reconstructable from $N^+(G)$.

Pf:

Frequency function:

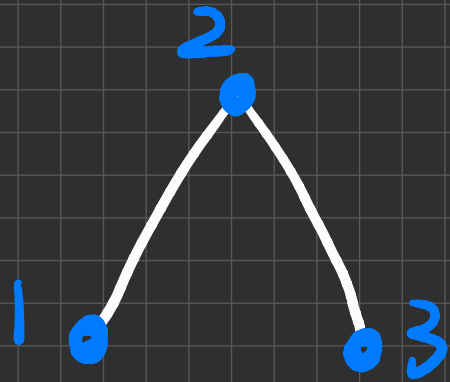
$$f_v : V(G) \rightarrow \mathbb{N}$$

$$f_v(u) := |\{N : \{u, v\} \subseteq N \in \mathcal{N}^*(G)\}|$$

Frequency function:

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$$\mathcal{N}^*(G) = \{\{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$$

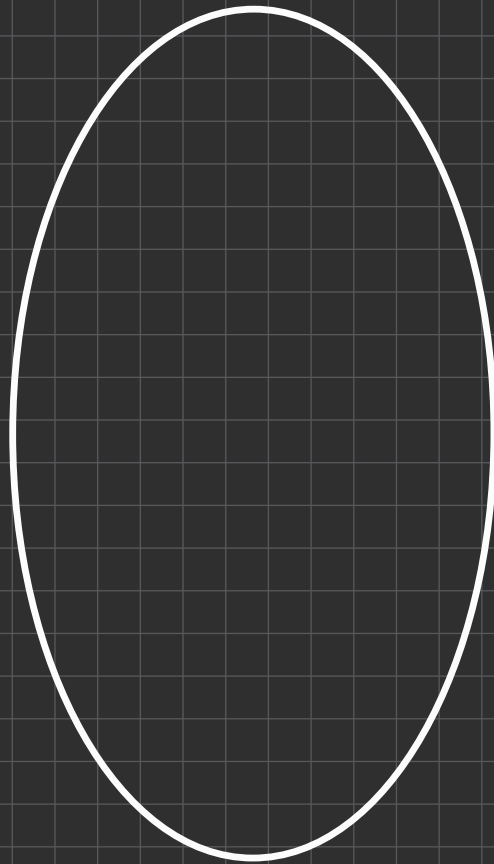
$$f_1(2) = 2$$

$$f_1(3) = 1$$

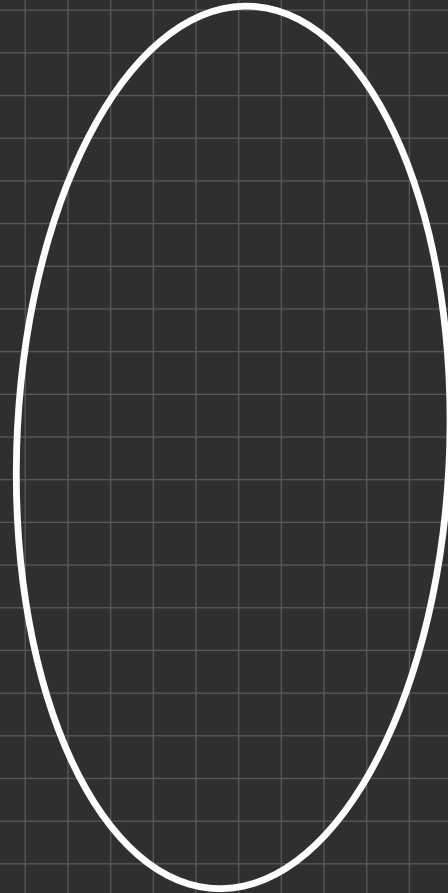
$$f_2(3) = 2$$

Fix a vertex v

v



$\text{dist} = 1$



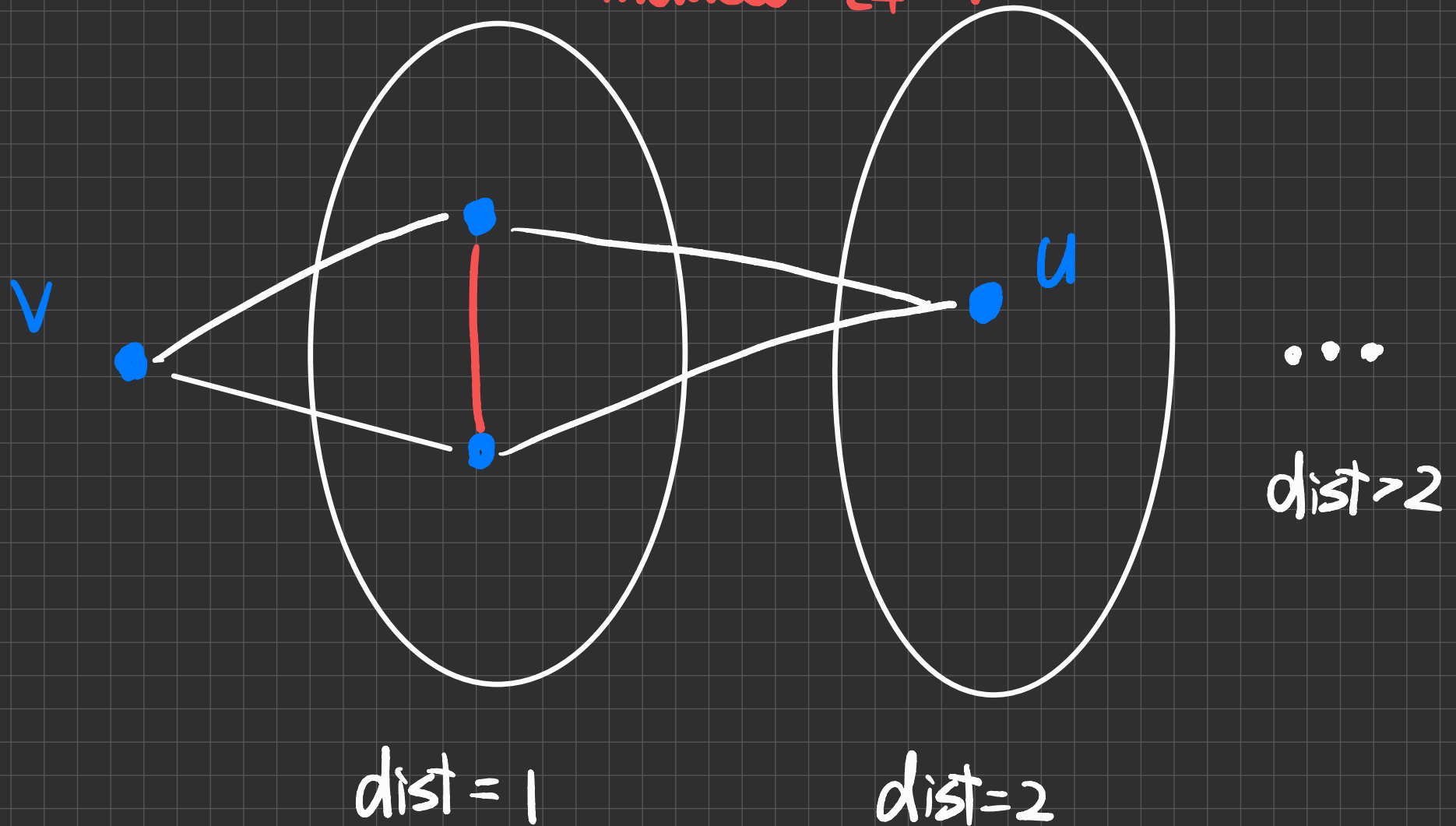
$\text{dist} = 2$

...

$\text{dist} > 2$

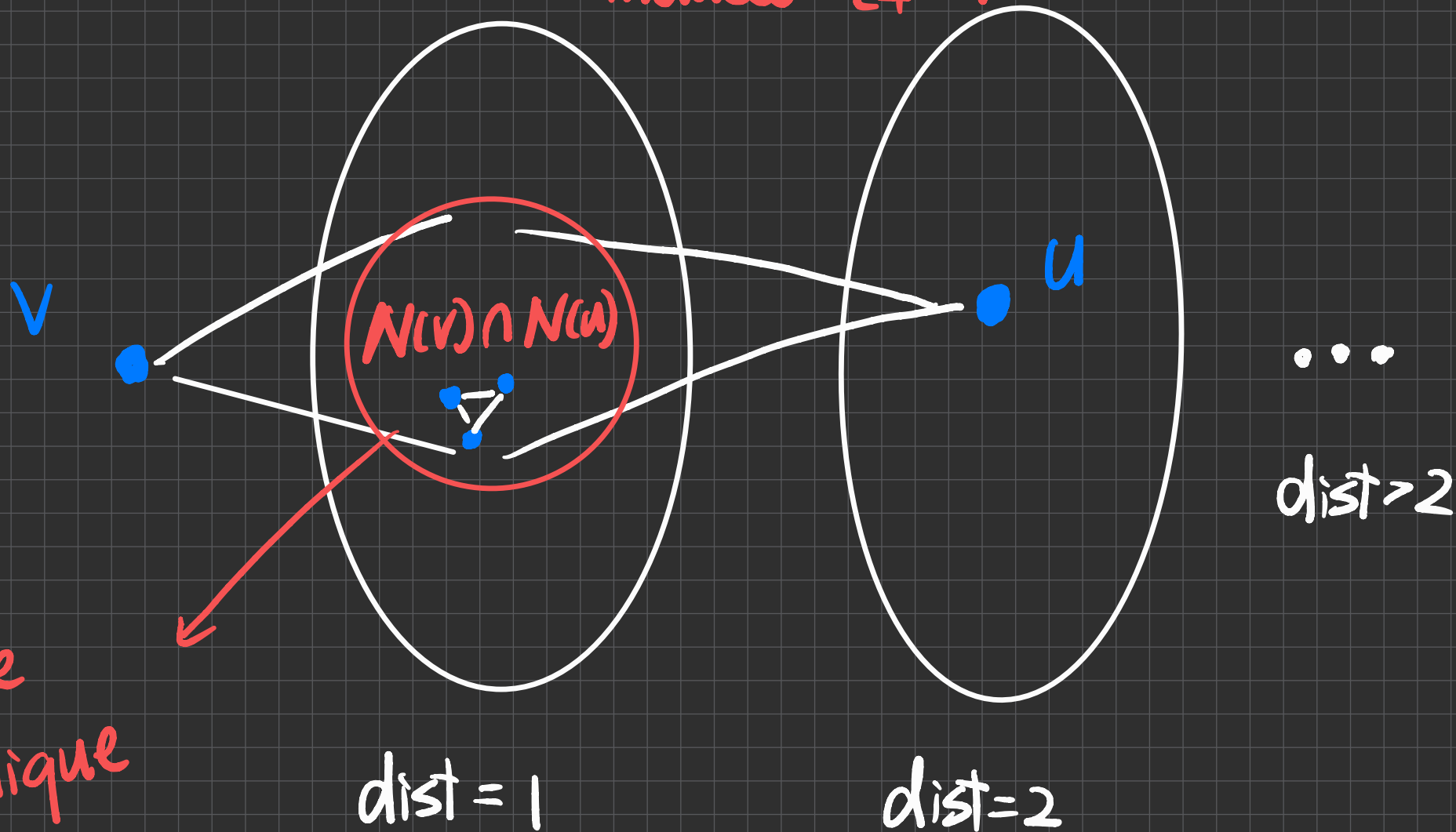
Fix a vertex v

induced- C_4 -free



Fix a vertex v

induced- C_4 -free



$x \in V(G) - v :$

- $\text{dist}(v, x) \geq 3$, if $f_v(x) = 0$

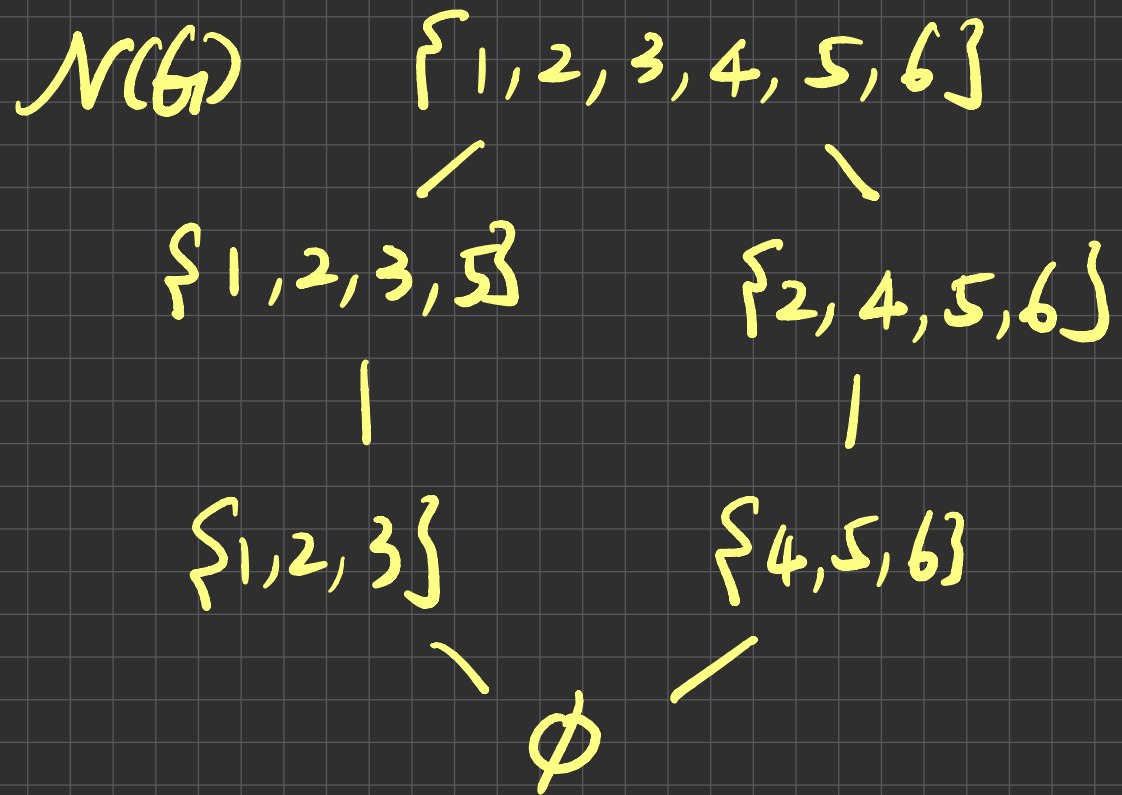
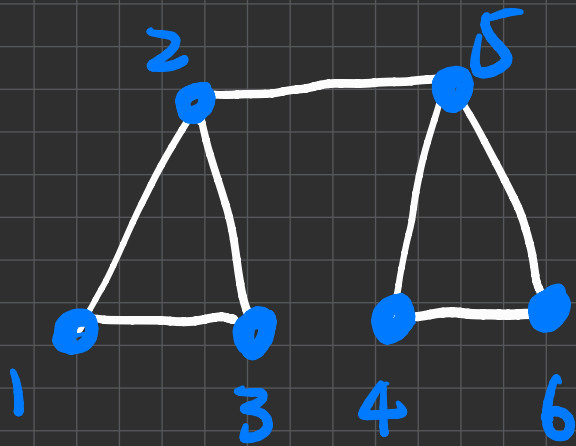
$x \in V(G) - v$:

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- $\text{dist}(v, x) = 2$, if $\forall N \supseteq \{v, x\}$, N contains at least $f_v(x) + 1$ vertices with frequency $> f(v, x)$

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- $\text{dist}(v, x) \geq 3$, if $f_v(x) = 0$
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- $\text{dist}(v, x) = 1$, if $\exists N \supseteq \{v, x\}$, N contains at most $f_v(x) - 1$ vertices with frequency $> f_v(x)$





$$N^*(G) = \int \{1, 2, 3\}^2, \{4, 5, 6\}^2, \{1, 2, 3, 5\}, \{2, 4, 5, 6\}$$

Union base of $\mathcal{N}(G)$:

$$\{N_1, N_2, \dots, N_r\} \quad \text{s.t.}$$

- $\mathcal{N}(G) = \left\{ \bigcup_{i \in I} N_i : I \subseteq [r] \right\}$

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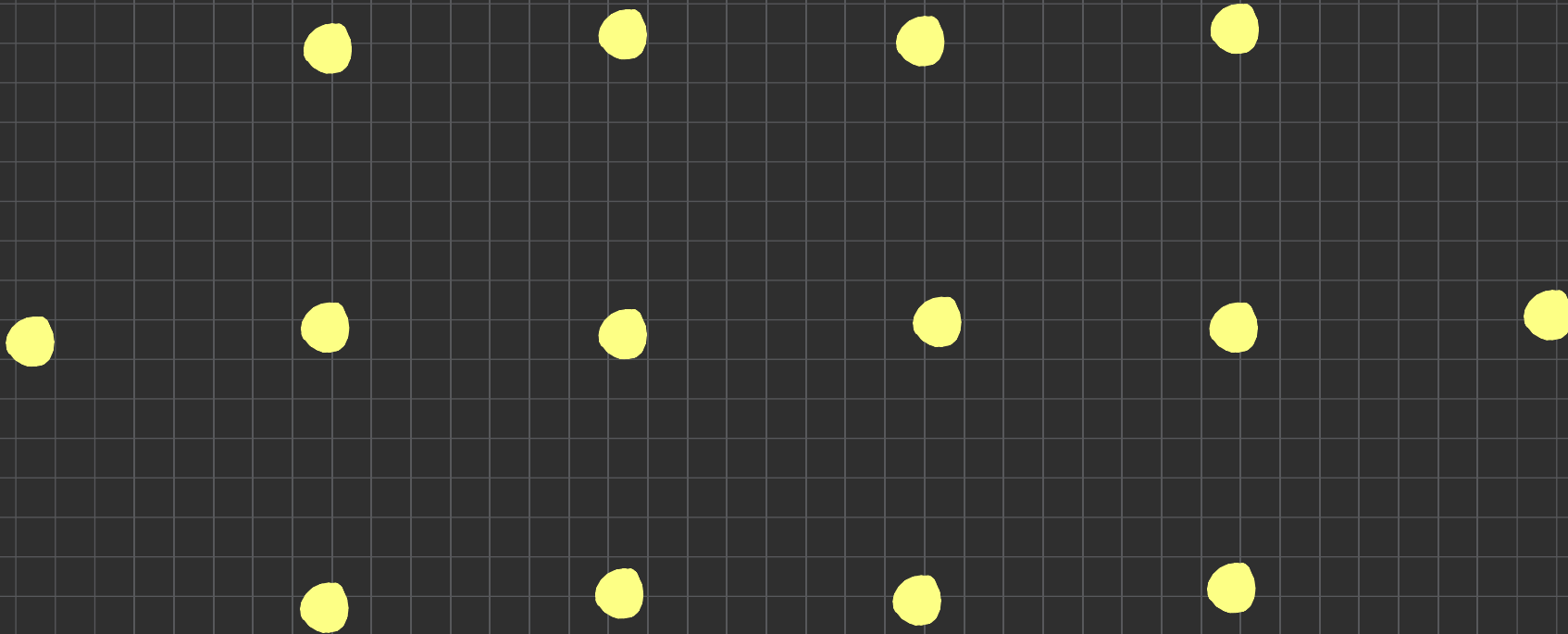
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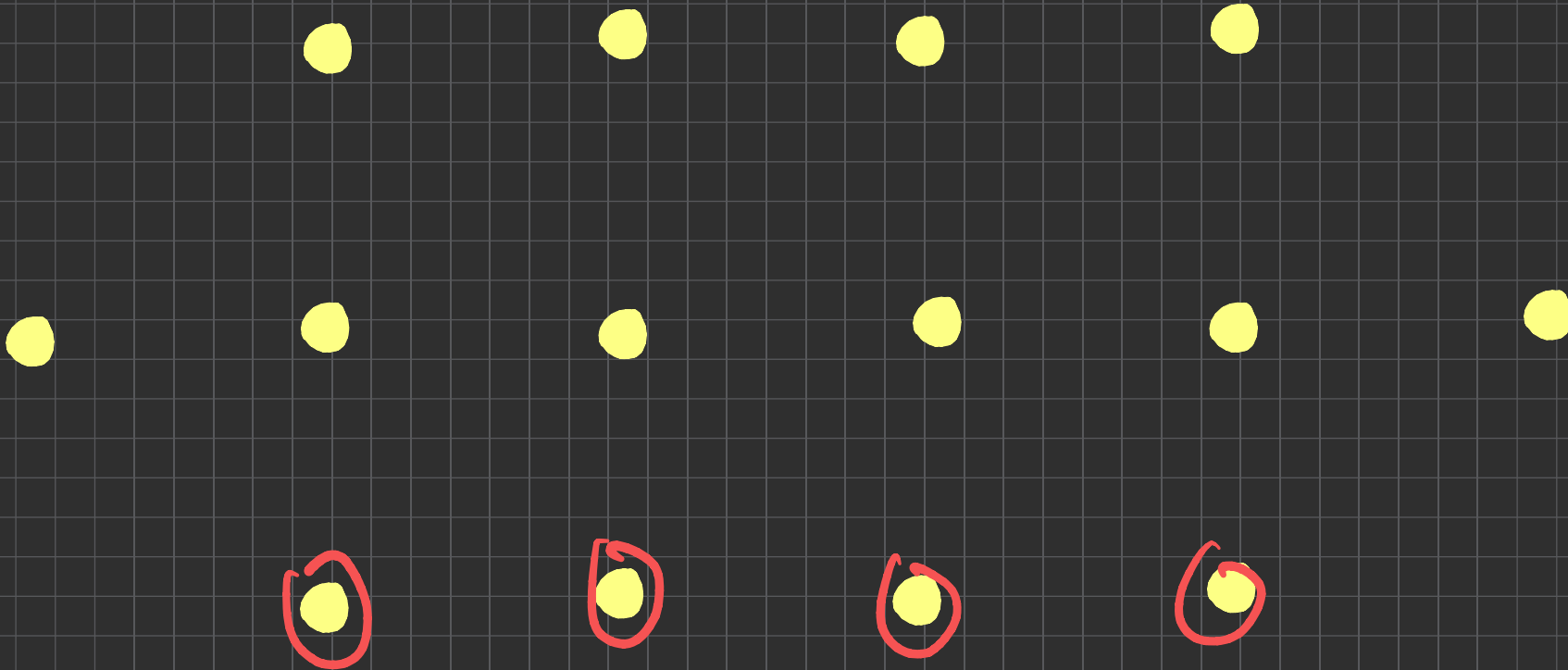
- $\mathcal{N}(G) = \left\{ \bigcup_{i \in I} N_i : I \subseteq [r] \right\}$
- for any $k \in [r]$, $N_k \neq \bigcup_{i \in I} N_i$, $\forall I \subseteq [r] - \{k\}$
 $N_i = N[v]$ for some v . $\{N[v_1], N[v_2], \dots, N[v_r]\}$

$V(G)$



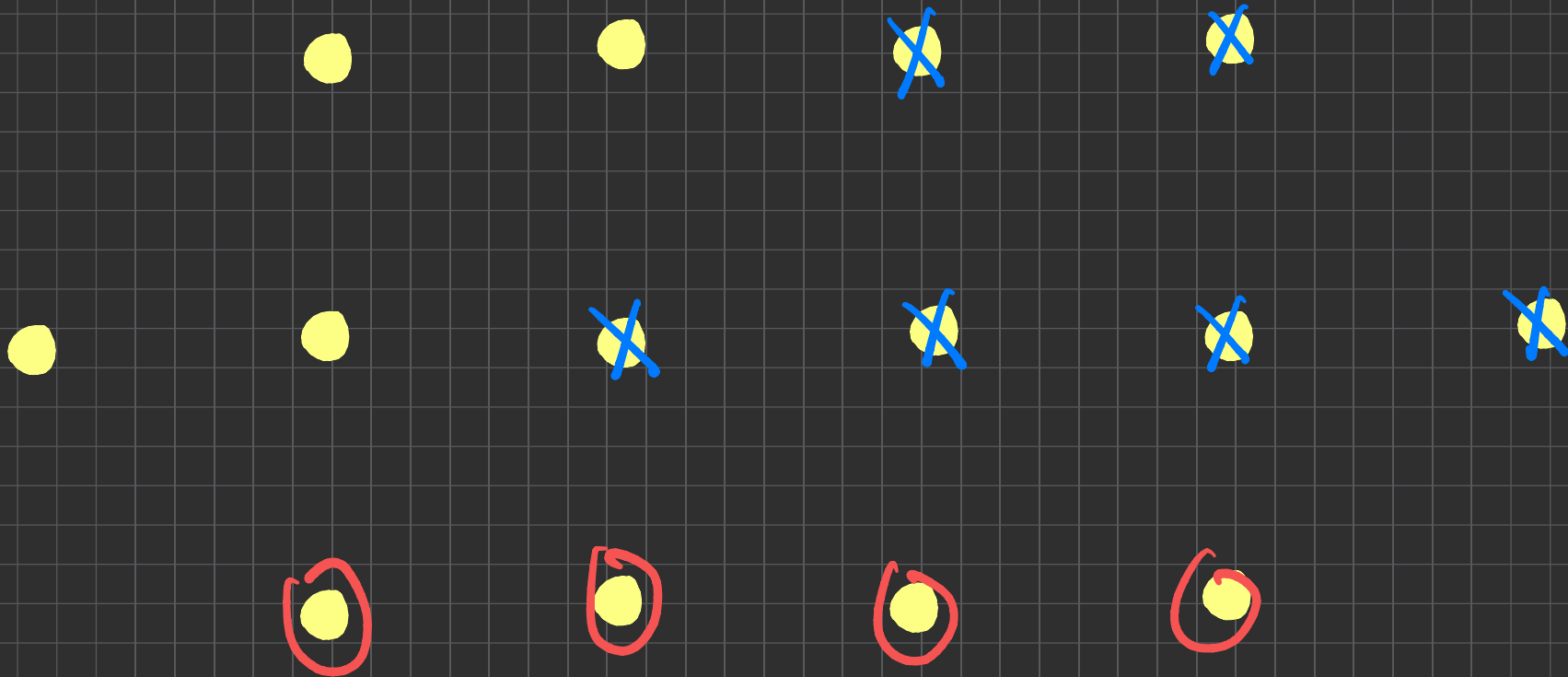
$\emptyset$

$V(G)$



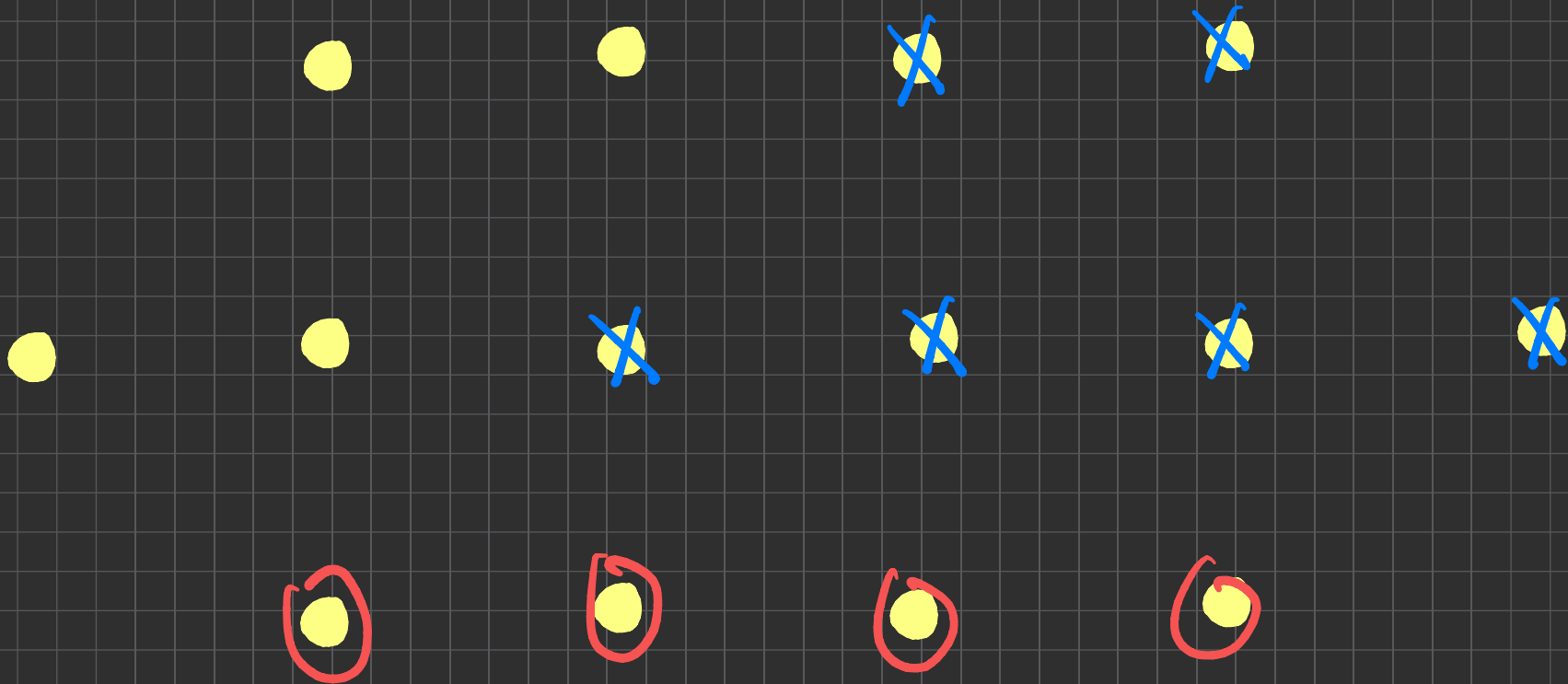
$\emptyset$

$V(G)$



$\emptyset$

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$\emptyset$

$V(G)$



$\emptyset$

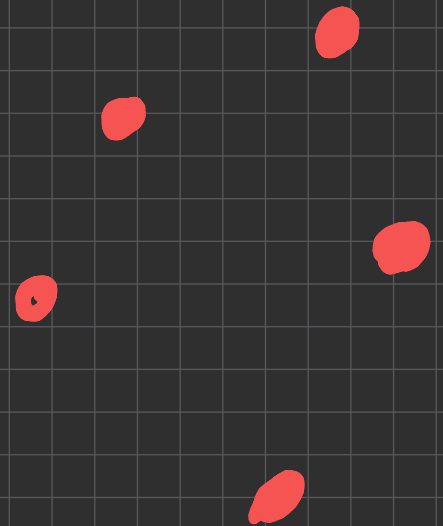
$V(G)$



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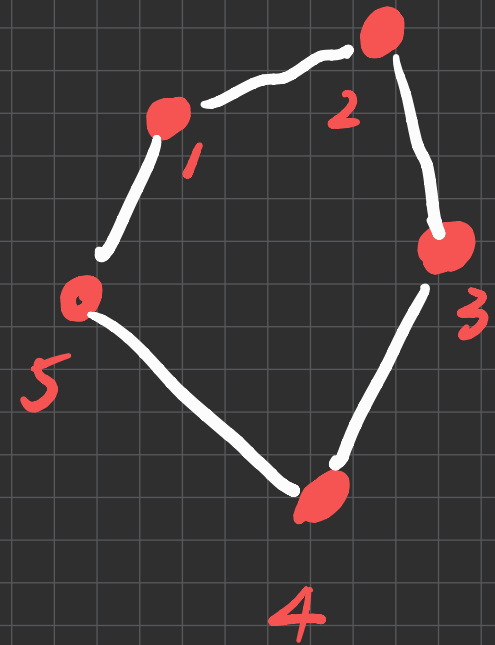
Reconstruction algorithm:

1. find union bases of $N(G)$



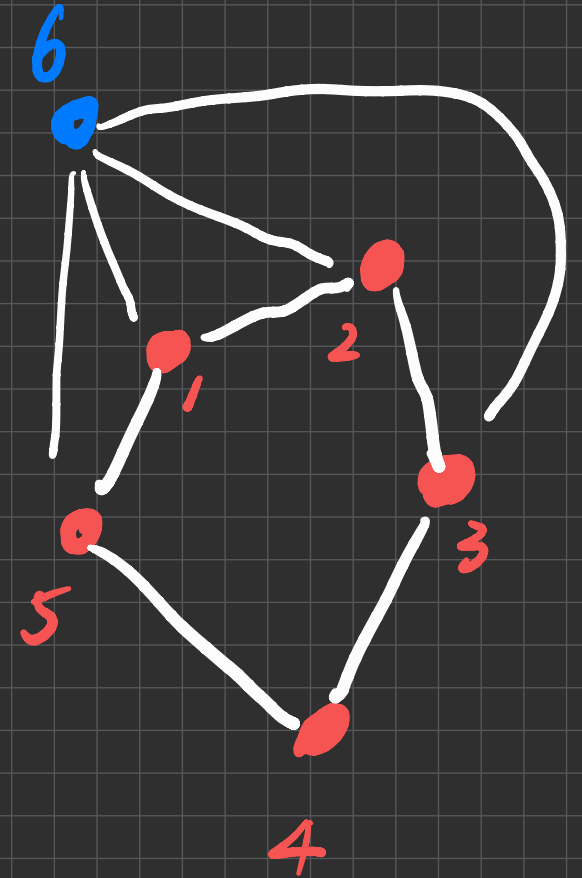
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2. apply lemma 1 on union bases
- 3.* recover the rest



Reconstruction algorithm:

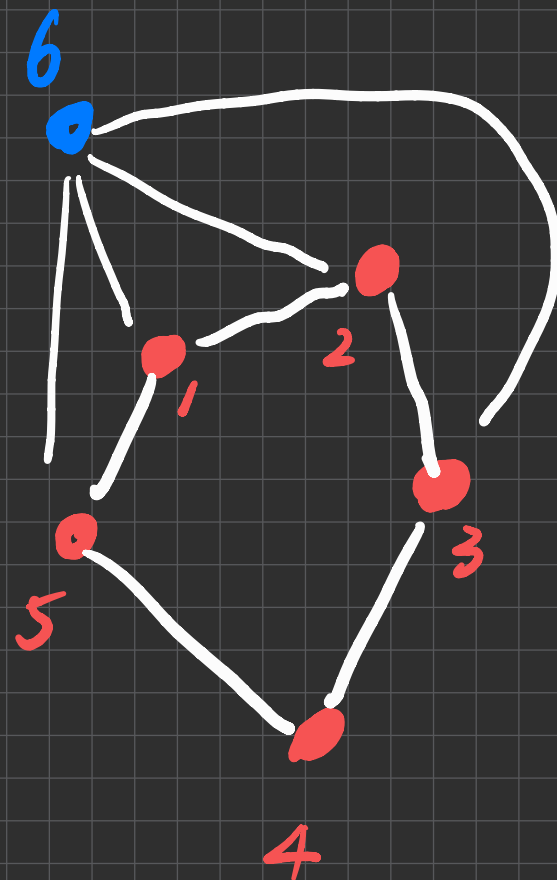
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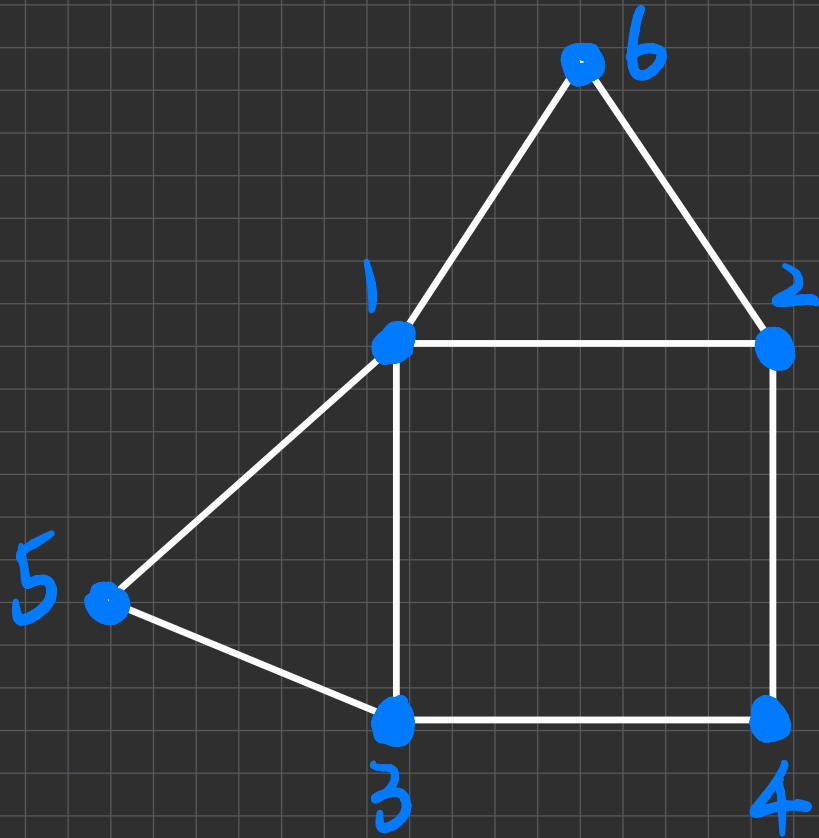
($N[x] = N[A]$ for some $A \subseteq \{v_1, v_2, \dots, v_r\}$)

Lemma. $N[A] = N[B]$ iff $A \cap N \neq \emptyset \Leftrightarrow B \cap N \neq \emptyset$
for all $N \in N(G)$.



induced- C_4 -free is "almost" the best condition

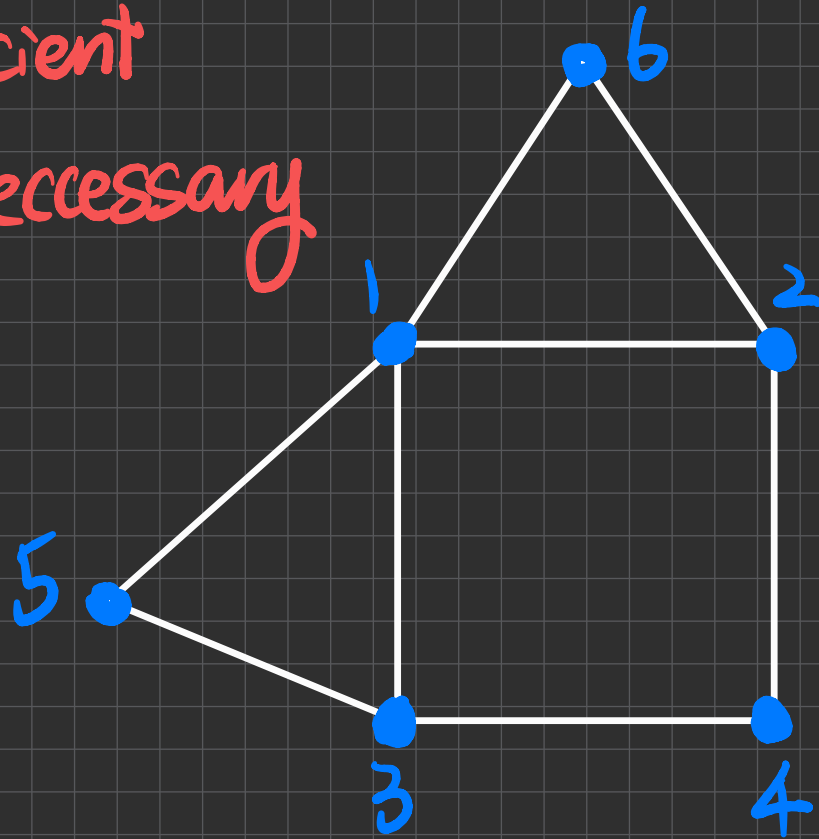
induced- C_4 -free is "almost" the best condition



is reconstructable

induced- C_4 -free is "almost" the best condition

- sufficient
- Not necessary



is reconstructable

In 2016, Hammack and Mullican gave a necessary and sufficient condition on graphs reconstructable from neighborhood multiset

$$\mathcal{N}^0(G) = \{N(x) : x \in V(G)\}.$$

Open Problems :

- What is a necessary and sufficient condition for our problem?

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- Given a collection $\mathcal{N}$ closed under taking Union. How do we know if $\exists G$ s.t. $\mathcal{N} = \mathcal{N}(G)$?

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- What is a necessary and sufficient condition for our problem?
- Given a collection $\mathcal{N}$ closed under taking Union. How do we know if $\exists G$ s.t $\mathcal{N} = \mathcal{N}(G)$?
- Can we recover $\mathcal{N}(G)$ from the deck of cards of G after labelling vertices?

Thank you !

