Extremal connectivity in graphs

Wayne Ge

LSU Combinatorics Seminar, February 2026

Let's start with a game

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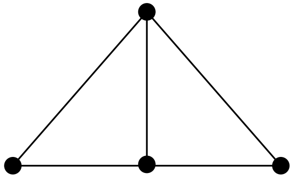
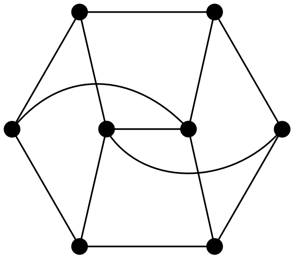
Can you find an edge whose deletion will not hurt 2-connectedness of the graph?



Let’s make it harder

Can you find an edge whose deletion will not hurt 3-connectedness of the graph?



minimally k-connected graphs

Definition

A k-connected graph G is **minimally k-connected** if $G \setminus e$ is not k-connected for all $e \in E(G)$.

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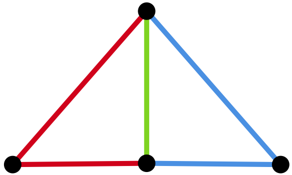

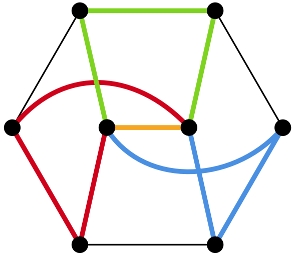

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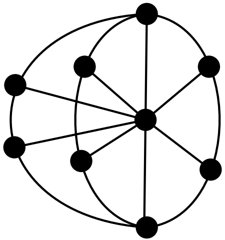
- When $n \geq 4$, the graph $K_{3,n}$ is m3c but not c3c.
- $W^+(3,3,3)$ is c3c but not m3c. (You can make find infinite examples of this kind)



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A graph G is **uniformly k-connected** if, for every pair of distinct vertices, there are exactly k internally disjoint paths connecting them.

-Obviously, being uniformly k-connected implies being minimally k-connected.
-It is not hard to prove that being uniformly k-connected also implies being critically k-connected.

super-minimally k-connected graphs

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Lemma

For all integers k exceeding one and all graphs G,

(i) *if G is uniformly k-connected, then G is super-minimally k-connected, and*

(ii) *if G is super-minimally k-connected, then G is both minimally and critically k-connected.*

Moreover, for all $k \geq 3$, neither of the converses holds.

Examples that might give you better intuition

When $k = 1$

- Uniformly 1-connected graphs are nontrivial trees.

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- Uniformly 2-connected graphs are cycles.
- Super-minimally 2-connected graph are also cycles.

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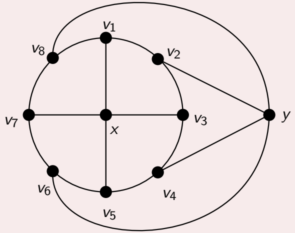

Figure: An alternating double wheel A_4.

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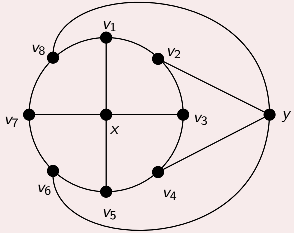

Figure: An alternating double wheel A_4.

When $k \geq 4$

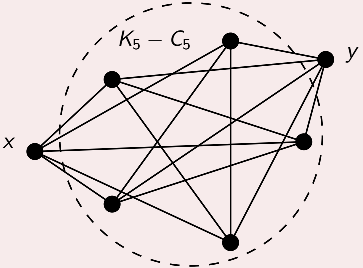

Figure: Q_5 is super-minimally 4-connected, but not uniformly 4-connected.

Hierarchy of 3-connected classes

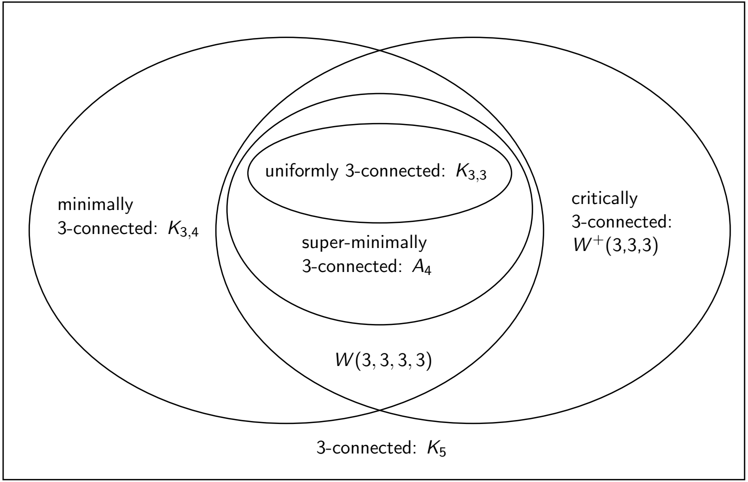

Figure: A Venn diagram of super-minimally, minimally, critically, and uniformly 3-connected graphs.

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- It cannot have many edges.

Part I: vertices of degree k

Theorem (Dirac 1967)

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Theorem (Mader 1972)

For all $k \geq 2$, a minimally k-connected graph G has at least $\frac{(k-1)|V(G)|+2k}{2k-1}$ vertices of degree k.

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Theorem (Göring, Hofmann, and Streicher 2022)

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Observation

A uniformly 2-connected or a super-minimally 2-connected graph G is a cycle and has exactly $|V(G)|$ vertices of degree two.

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Theorem (G. 2025+)

A super-minimally 3-connected graph G has at least $\frac{|V(G)|+3}{2}$ vertices of degree three.

The structural of minimally 3-connected graphs

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If G is a minimally 3-connected graph, then every cycle meets at least two vertices of degree three of G.

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Let V_+ be the set of vertices of degree at least four. Then $G[V_+]$ is a forest F.

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If G is a minimally 3-connected graph and e is an edge of order at least four, then G/e is minimally 3-connected.

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Halin's proof sketch

$$\begin{aligned}
\sum_{v \in V_+} \deg(v) - 2|E(F)| &\leq 3|V_3| \\
4|V_+| - 2|E(F)| &\leq 3|V_3| \\
4|V_+| - 2(|V_+| - \#\text{components of } F) &\leq 3|V_3| \\
2|V(G)| + 2\#\text{components of } F &\leq 5|V_3|
\end{aligned}$$

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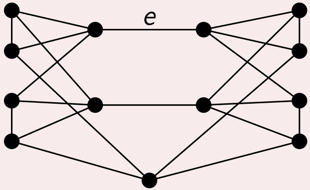
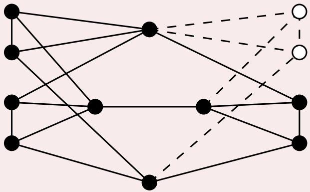
Some major difficulties

- When G is sm3c and e is an edge of order at least four, G/e is not necessarily sm3c.



(a). G



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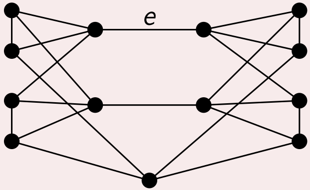

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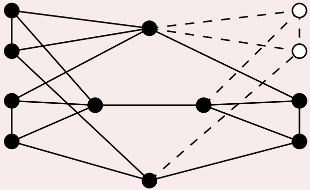

(b). G/e

- We don't know what happens on the $G[V_3]$ side. Assuming them each contributes three edges to the "middle" will not improve Halin's bound.

Our approach

Lemma

Let G be a minimally and critically 3-connected graph and F be the subgraph induced by the vertices of degree greater than three. Then $G/E(F)$ is both minimally and critically 3-connected. In particular, $G/E(F)$ is simple.

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What we actually proved.

Theorem

If G is both minimally 3-connected and critically 3-connected, then

$$v_3(G) \geq \frac{v(G)+3}{2}.$$

Major steps

Step 1

Each vertex in V_3 may contribute $0, 1, 2$ or 3 edges to the "middle". (Here t_i is the number of degree three vertex that contributes i edges. Clearly $t_0 + t_1 + t_2 + t_3 = |V_3|$.)

$$\sum_{v \in V(F)} d_G(v) - 2|E(F)| = \sum_{i \in \{0,1,2,3\}} i \cdot t_i.$$

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$$\sum_{v \in V(F)} d_G(v) - 2|E(F)| = \sum_{i \in \{0,1,2,3\}} i \cdot t_i.$$

Step 2

Contract all the edges of order at least four to get G', which is also minimally and critically 3-connected. Using similar trick as Halin's proof, we get

$$2v_+(G) + 2v_+(G') \leq \sum_{i \in \{0,1,2,3\}} i \cdot t_i. \tag{1}$$

Major steps

Step 3

From G', we delete every degree-3 vertex if it does not contribute three edges to the "middle". The resulting graph is bipartite. The intuition is that t_3 cannot be too large comparing to $v_+(G')$, otherwise there is a deletable vertex.

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Step 4

Combining

$$2v_+(G) + 2v_+(G') \leq \sum_{i \in \{0,1,2,3\}} i \cdot t_i. \tag{1}$$

$$t_3 \leq 2v_+(G') - 5 \tag{2}$$

$\square$

Extremal examples

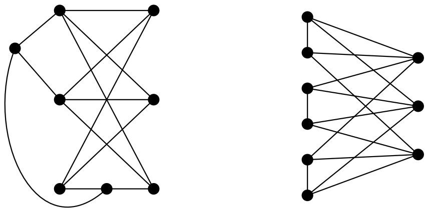

Figure: Two small extremal examples.

Extremal examples

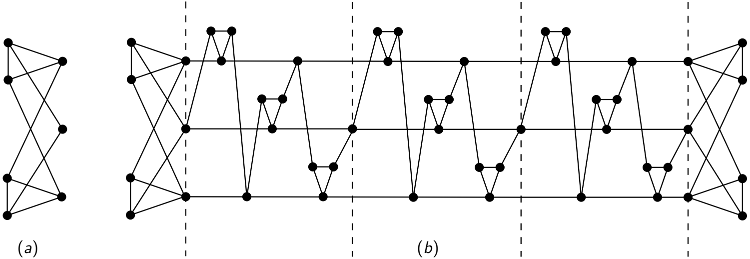

Figure: (a) A buckle and (b) a belt of length three.

Part II: number of edges

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Theorem (Dirac 1967)

If G is a minimally 2-connected graph with $|V(G)| \geq 4$, then

$$|E(G)| \leq 2|V(G)| - 4.$$

Moreover, equality holds if and only if G is isomorphic to $K_{2,|V(G)|-2}$.

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Theorem (Halin 1969)

If G is a minimally 3-connected graph with $|V(G)| \geq 7$, then

$$|E(G)| \leq 3|V(G)| - 9.$$

Moreover, if $|V(G)| \geq 8$, equality holds if and only if G is isomorphic to $K_{3,|V(G)|-3}$.

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Theorem (Xu 2024)

If G is a uniformly 3-connected graph, then

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Moreover, equality is attained if and only if G is a wheel with at least three spokes.

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Observation

If G is a uniformly or super-minimally 2-connected graph, then...

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This result implies (Xu 2024).

Some reduction operations

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Cleaving

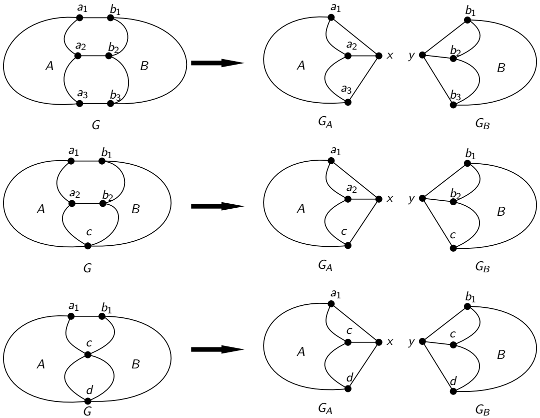

Figure: Cleaving G into G_A and G_B.

Some reduction operations

Enhanced deletion $G\backslash\backslash e$ **(Basically** $\mathrm{co}(M \setminus e)$**)**

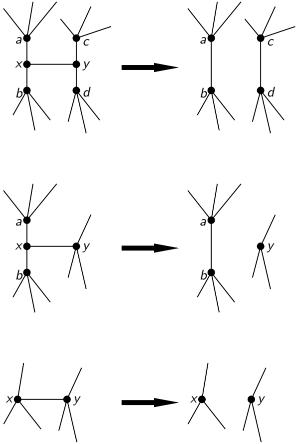

Figure: $G\backslash\backslash xy$ in different cases.

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If G is super-minimally 3-connected and G/e is not 3-connected, then $G\backslash\backslash e$ is super-minimally 3-connected.

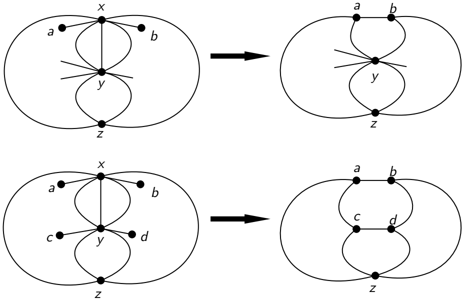

Figure: Two possible cases of $G\backslash\backslash xy$.

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Using these reduction operations, we may impose very strong structural properties on the minimal counterexample.

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The original proof is very technical and hence omitted here. There is a simpler, and more general proof.

Part III: Constructive characterizations

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“Every graph G in $\mathcal{G}$ can be built from a member S in $\mathcal{S}$, by iteratively applying operations in $\mathcal{O}$. Every graph in this construction chain is also in $\mathcal{G}$.”

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Some bridging operations

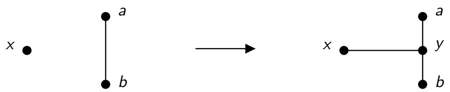
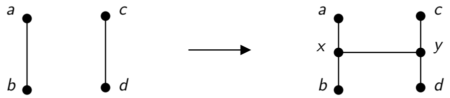


Figure: A vertex-to-edge bridging.



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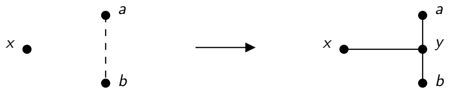

Figure: A vertex-to-non-edge bridging.

Minimally and uniformly 3-connected graphs

Theorem (Dawes 1984)

Every minimally 3-connected graph G can be obtained from K_4 by iteratively applying vertex-to-edge, edge-to-edge, and vertex-to-non-edge bridging on some "compatible" sets.

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Conjecture

Every super-minimally 3-connected graph G can be obtained from K_4 by iteratively applying vertex-to-edge, and edge-to-edge bridging on some "pseudo-compatible" sets.

Future directions

- **Super-minimally k-connected graphs, for $k \geq 4$.**

Conjecture (G. 2025+)

For every integer $k \geq 3$, there are constants a_k and b_k such that

(i) every super-minimally k-connected graph has at least $a_k|V(G)|$ degree-k vertices,

(ii) every uniformly k-connected graph has at least $b_k|V(G)|$ degree-k vertices, and

(iii) $\frac{k-1}{2k-1} < a_k < b_k$.

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Conjecture (Oxley 2025)

If M is a binary matroid without a 3-connected restriction of size at least four, then its critical exponent $c(M;2) \leq 2$.

Summary

Class	Min. no. of deg-3 vertices	Max. no. of edges
Minimally 3-conn.	$\frac{2n+6}{5}$	$3n-9$
Super-min. 3-conn.	$\frac{n+3}{2}$	$2n-2$
Uniformly 3-conn.	$\frac{2n+2}{3}$	$2n-2$

Table: Bounds for three families of 3-connected graphs ($n = |V(G)|$).